Extremal Bounds for Functions of Bounded Turning

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Abstract
In this article, we determined the upper and lower bounds for functions having the properties of bounded turning by using the convexity techniques. Moreover, the bounds of the partial sums are estimated near the origin.

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1 Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots.$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \quad (1)$$

For $0 \leq \mu < 1$, let $B(\mu)$ denote the class of functions $f$ of the form (1) so that $\Re\{f'\} > \mu \in U$. The functions in $B(\mu)$ are called functions of bounded turning. By the Nashiro-Warschowskii Theorem, the functions in $B(\mu)$ are univalent and also close-to-convex in $U$. Recently, this class is generalized and studied by many authors (see [2,3,10-12]).
Let $f$ be analytic in $U$, $g$ analytic and univalent in $U$ and $f(0) = g(0)$. Then, by the symbol $f(z) \prec g(z)$ (if subordinate to $g$) in $U$, we shall mean $f(U) \subset g(U)$.

Let $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$ then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination if $p \prec q$. If $p$ and $\phi(p(z)), zp'(z))$ are univalent in $U$ and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$ then $p$ is called a solution of the differential superordination. An analytic function $q$ is called subordinant of the solution of the differential superordination if $q \prec p$. We use the notation $s(g) = \{f \in \mathcal{H}(U) : f \prec g\}$, for details (see [8]).

Let $X$ be a locally convex linear topological space. For a subset $U \subset X$ the closed convex hull of $U$ is defined as the intersection of all closed convex sets containing $U$ and will be denoted by $co(U)$. If $U \subset V \subset X$ then $U$ is called an extremal subset of $V$ provided that whenever $u = tx + (1-t)y$ where $u \in U, x, y \in V$ and $t \in (0, 1)$ then $x, y \in U$. An extremal subset of $U$ consisting of one point is called an extreme point of $U$. The set of the extreme points of $U$ will be denoted by $\mathcal{E}(U)$.

**Remark 1** If $L : \mathcal{H}(U) \rightarrow \mathcal{H}(U)$ is an invertible linear map and $F \subset \mathcal{H}(U)$ is a compact subset, then $L(co(F)) = co(L(F))$ and the set $\mathcal{E}(co(F))$ is in one-to-one correspondence with $\mathcal{E}(co(F))$.

In the present paper, we introduce functions $f \in \mathcal{A}$ in the class $\mathcal{B}_m(\mu), m \in \mathbb{N}, \mu > 0$ which satisfy the condition

$$\Re \left\{ \sqrt[m]{f'(z)} \right\} > \mu. \quad (2)$$

It is clear that $\mathcal{B}_1(\mu) \equiv B(\mu)$.

In order to obtain our results, we need the following techniques.

**Lemma 1** [5] Suppose that $F_\alpha$ is defined by the equality

$$F_\alpha(z) = \left( \frac{1 + cz}{1 - z} \right)^\alpha, \quad (|c| \leq 1, c \neq -1).$$

If $\alpha \geq 1$ then $co(s(F_\alpha))$ consists of all functions in $\mathcal{H}(U)$ represented by

$$f(z) = \int_0^{2\pi} \left( \frac{1 + cz e^{-it}}{1 - ze^{-it}} \right)^\alpha d\mu(t),$$

where $\mu$ is a positive measure on $[0, 2\pi]$ having the property $\mu([0, 2\pi]) = 1$ and

$$\mathcal{E}(co(s(F_\alpha))) = \left\{ \left. \frac{1 + cz e^{-it}}{1 - ze^{-it}} \right| t \in [0, 2\pi] \right\}.$$
Lemma 2 [5] If $J : \mathcal{H}(U) \rightarrow \mathbb{R}$ is a real-valued, continuous convex functional and $\mathcal{F}$ is a compact subset of $\mathcal{H}(U)$, then

$$\max\{J(f) : f \in \text{co}(\mathcal{F})\} = \max\{J(f) : f \in \mathcal{F}\} = \max\{J(f) : f \in \mathcal{E}(\text{co}(\mathcal{F}))\}.$$  

In the particular case if $J$ is a linear map then we also have:

$$\min\{J(f) : f \in \text{co}(\mathcal{F})\} = \min\{J(f) : f \in \mathcal{F}\} = \min\{J(f) : f \in \mathcal{E}(\text{co}(\mathcal{F}))\}.$$  

Lemma 3 [7] For $z \in U$ we have

$$\Re\left\{ \sum_{n=1}^{j} \frac{z^n}{n+2} \right\} > -\frac{1}{3}, \quad (z \in U).$$

2 Main Results

We have the following results:

Theorem 1 Let $f \in \mathcal{B}_m(\mu), m \in \mathbb{N}, \mu > 0$. Then

$$1 + \inf_{z \in U} \Re \left( \sum_{n=1}^{\infty} \sum_{k=0}^{m} C_k^m C_{m+n-k-1}^{m-1} z^n \right) < \Re \left( f'(z) \right)$$

$$< 1 + \sup_{z \in U} \Re \left( \sum_{n=1}^{\infty} \sum_{k=0}^{m} C_k^m C_{m+n-k-1}^{m-1} z^n \right).$$

The result is sharp.

Proof Since $f \in \mathcal{B}_m(\mu)$ then we have

$$\sqrt[2]{f'(z)} < \frac{1+z}{1-z}$$

which is equivalent to

$$f'(z) < \left( \frac{1+z}{1-z} \right)^m.$$  

In view of Lemma 1,

$$f'(z) = \int_{0}^{2\pi} \left( \frac{1+ze^{-it}}{1-ze^{-it}} \right)^{\alpha} d\mu(t) := h(z), \quad (4)$$
where \( \mu \) is a positive measure on \([0, 2\pi]\) having the property \( \mu([0, 2\pi]) = 1 \). Assume that

\[
f'(z) = 1 + \sum_{n=2}^{\infty} n a_n z^{n-1}
\]

\[
= 1 + \sum_{n=1}^{\infty} (n + 1) a_{n+1} z^n
\]

\[
:= 1 + \sum_{n=1}^{\infty} b_n z^n,
\]

where \( b_n := (n + 1)a_{n+1} \).

On the other hand we observe that

\[
\int_0^{2\pi} \left( \frac{1 + ze^{-it}}{1 - ze^{-it}} \right)^\alpha d\mu(t) = 1 + \sum_{n=1}^{\infty} \left[ \sum_{k=0}^{n} C_m^k C_{m+n-k-1}^{m-1} \right] z^n \int_0^{2\pi} e^{-int} d\mu(t)
\]

such that \( C_i^j = 0 \), \( i > j \) and

\[
\sum_{k=0}^{n} C_m^k C_{m+n-k-1}^{m-1} = \sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1}.
\]

Thus (6) and (7) imply

\[
1 + \sum_{n=1}^{\infty} b_n z^n = 1 + \sum_{n=1}^{\infty} \left[ \sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1} \right] z^n \int_0^{2\pi} e^{-int} d\mu(t)
\]

and

\[
b_n = \left( \sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1} \right) \int_0^{2\pi} e^{-int} d\mu(t).
\]

Consequently,

\[
f'(z) = 1 + \sum_{n=1}^{\infty} \left( \sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1} \right) z^n \int_0^{2\pi} e^{-int} d\mu(t).
\]

If

\[
\mathcal{H} = \left\{ h \in \mathcal{H}(U) : h(z) = \int_0^{2\pi} \left( \frac{1 + ze^{-it}}{1 - ze^{-it}} \right)^\alpha d\mu(t), \quad \mu([0, 2\pi]) = 1 \right\}
\]

and

\[
\mathcal{Q} = \left\{ q \in \mathcal{H}(U) : q(z) = f'(z), \quad f \in \mathcal{B}_m(\mu) \right\}
\]
then the correspondence

\[ L : \mathcal{H} \rightarrow \mathcal{Q}, L(h) = q(z) \]

defines an invertible linear map and according to Remark 1, the extreme points of the class \( \mathcal{Q} \) are

\[ q_t(z) = 1 + \sum_{n=1}^{\infty} \left( \sum_{k=0}^{m} C_m^k C_{m+n-k-1} \right) z^n e^{-int}, \quad (z \in U, t \in [0, 2\pi]). \]

Hence Lemma 2 implies the assertion of Theorem 1.

**Theorem 2** For \( f \in A \), let

\[ f_k(z) = z + \sum_{n=2}^{k} a_n z^n, \quad (z \in U) \]

be its partial sum. And let

\[ F_k(z) = z + \sum_{n=2}^{k} \frac{2}{n+1} a_n z^n, \quad (z \in U) \]

be the partial sum of the Libera integral operator

\[ F(z) = \frac{2}{z} \int_0^z f(\zeta) d\zeta = z + \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^n. \]

Then

\[ \frac{2}{3} < \Re\left( f'_k(z) \right) \leq 1, \quad (z \to 0) \]

and

\[ \frac{1}{3} < \Re\left( F'_k(z) \right) \leq 1, \quad (z \to 0). \]

**Proof** By applying Lemma 3.

Next we pose upper and lower bounds for functions \( f \in B(0) \).

**Theorem 3** Let \( 0 < \alpha \leq 2 \) be a real number. If \( f \in A, f'(z) \neq 0, z \in U \), satisfies the differential subordination

\[ (1 - \alpha)f'(z) + \alpha(1 + \frac{zf''(z)}{f'(z)}) \prec 1 + 2(1 - \alpha) \frac{z}{1 - z} + 2\alpha \frac{z}{1 - z^2}. \]
Then
\[ 1 + \inf_{z \in U} \Re\left( \sum_{n=1}^{\infty} 2z^n \right) < \Re\left( f'(z) \right) < 1 + \sup_{z \in U} \Re\left( \sum_{n=1}^{\infty} 2z^n \right). \] (8)
The result is sharp.

**Proof** By applying [Corollary 3.4, 9] we have \( \Re(f'(z)) > 0 \) and consequently by Theorem 1, we obtain the desired assertion.

**Theorem 4** Let \( 0 < \alpha < 1 \) be a real number. If \( f \in A, f'(z) \neq 0, z \in U \), satisfies the differential inequality
\[ \Re\left\{ (1 - \alpha)f'(z) + \alpha(1 + \frac{zf''(z)}{f'(z)}) \right\} > \alpha. \]
Then
\[ 1 + \inf_{z \in U} \Re\left( \sum_{n=1}^{\infty} 2z^n \right) < \Re\left( f'(z) \right) < 1 + \sup_{z \in U} \Re\left( \sum_{n=1}^{\infty} 2z^n \right). \] (9)
The result is sharp.

**Proof** By using [Corollary 3.6, 9] we pose \( \Re(f'(z)) > 0 \) and hence in view of Theorem 1, we have the desired assertion.

**Theorem 5** Let \( \alpha \neq 0 \) be a complex number. Let \( q \neq 0 \) be univalent in \( U \) such that
\[ \Re\left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > \max\{0, \Re\left( \frac{\alpha - 1}{\alpha} q(z) \right) \}. \]
If \( f \in A, f'(z) \neq 0, z \in U \), satisfies the differential subordination
\[ (1 - \alpha)[f'(z) - 1] + \alpha\left( \frac{zf''(z)}{f'(z)} \right) < (1 - \alpha)[q(z) - 1] + \alpha\frac{zq'(z)}{q(z)}. \]
Then
\[ 1 + \inf_{z \in U} \Re\left( \sum_{n=1}^{\infty} 2z^n \right) < \Re\left( f'(z) \right) < 1 + \sup_{z \in U} \Re\left( \sum_{n=1}^{\infty} 2z^n \right). \] (10)
The result is sharp.

**Proof** In virtue of [Theorem 3.1, 9] and Theorem 1, we have the desired assertion.

For extra reading on partial sums and bounded turning problems, see for examples in ([1,4,6]).

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**References**


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