Dynamical Behavior of a Chemostat Model Concerning Impulsive State Feedback Control Strategy

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Abstract

In this paper, a chemostat model for concerning impulsive state feedback control strategy is proposed and investigated. We obtained sufficient conditions of global asymptotical stability of the system without impulsive state feedback control. And also did we obtain the system with impulsive state feedback control may have order one periodic solution, as well as, the sufficient conditions for existence and stability of order one solution is gotten for some cases.

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1. Introduction

The chemostat is a simple and well-adopted laboratory apparatus which used to study bacterial metabolism, plasmid stability, and population genetics. The advantages are obvious, because certain of the biological parameter assumed to influence the outcomes can be controlled by the experiences. Many papers have investigated the mathematical models on the culture of microorganisms. For example, mathematical and experiment [3] models exhibit the competitive exclusion principle by which only species survives. Tang[6] analysis the model for pest management. Biological dynamic is shown in [2]. Li[4]
analysis the turbidostate model. In the process of bio-reacts, variations such as temperature, dissolved oxygen content can affect the growth and reproduction of the microorganisms. In some cases, regulating the bio-reacts is necessary.

2. Model and preliminaries

2.1 The model

Let $s, x$ denote the concentration of substrate and the concentration of microorganism in the chemostat at time $t$, respectively. In order to the model is significance, we have $0 < s < 1$ and $0 < x < 1$. We could have the following model to describe the continuously culturing microorganism in a chemosate.

$$
\begin{align*}
\dot{s}(t) &= Q(s^0 - s) - \frac{u_m s}{(k_m + s)(A + Bs)} x, \\
\dot{x}(t) &= \frac{u_m s}{k_m + s} x - Q x, \\
\Delta s(t) &= -E_1 s, \\
\Delta x(t) &= -E_1 x,
\end{align*}
$$

(2.1)

Where $Q$ is the dilution rate of the chemostat. $s^0$ is the concentration of the input substrate. $\delta$ is the yield. $u_m > 0$ is maximal specific growth rate of the microorganisms. $k_m$ is the saturation constant. The yield $\delta$ of the chemostat is controlled by setting $\delta = A + Bs$ (in [1]), where $A > 0, B > 0$. The system (2.1) can be modified as follows by introducing the impulsive state feedback control:

$$
\begin{align*}
\dot{s}(t) &= Q(s^0 - s) - \frac{u_m s}{(k_m + s)(A + Bs)} x, \\
\dot{x}(t) &= \frac{u_m s}{k_m + s} x - Q x, \\
\Delta s(t) &= -E_1 s, \\
\Delta x(t) &= -E_1 x, \\
\end{align*}
$$

$x < x_m,$

(2.2)

$$
\begin{align*}
\dot{s}(t) &= Q(1 - S) - \frac{mS}{(a + dS)(c + S)} x, \\
\dot{x}(t) &= \frac{u_m s}{k_m + s} x - Q x, \\
\Delta s(t) &= -bs, \\
\Delta x(t) &= -bx, \\
\end{align*}
$$

$x = h,$

(2.3)

Where $0 < E_1 < 1$ is constant and is also the part of the concentration of microorganism that decreases due to the feedback control when the concentration $x$ reaches $x_m$. For simplicity, we non-dimensionalize system (2.2) with the following scaling: $S \rightarrow \frac{s^0}{s}$. By the scaling, we can obtain the form:

$$
\begin{align*}
\dot{S}(t) &= Q(1 - S) - \frac{mS}{(a + dS)(c + S)} x, \\
\dot{x}(t) &= \frac{u_m s}{k_m + s} x - Q x, \\
\Delta s(t) &= -bs, \\
\Delta x(t) &= -bx, \\
\end{align*}
$$

$x = h,$

(2.3)

Where $\frac{A}{(p')^x} = a; Bs^0 = d; \frac{k_m}{p'} = c; x_m = h; u_m = m; x_m = h; E_1 = b.$

2.2 Some important preliminaries
Definition 2.2.1 A triple \((X, \pi, R_+)\) is said to be a semi-dynamical system if \(X\) is metric space, \(R_+\) is the set of all non-negative reals and \(\pi: X \times R_+ \rightarrow X\) is a continuous function such that: (1) \(\pi(x, t) = x\) for all \(x \in R\), (2) \(\pi(x + s, t) = \pi(x, t + s)\) for all \(x \in R\) and \(t, s \in R_+\).

Definition 2.2.2 An impulsive semi-dynamical system \((X, \pi : M, I)\) consists of a semi-dynamical system \((X, \pi)\) together with a nonempty closed subset \(M\) of \(X\) and a continuous function \(I : M \rightarrow X\) such that the following properties hold: (1) No point \(x \in X\) is a limit point of \(M(x)\), (2) \([t]\{G(x, t) \cap M\} \neq \emptyset\) is a closed subset of \(R_+\).

We write \(N = I(M) = \{y \in X | y = I(x), x \in M\}\) and for any \(x \in X\), \(I(x) = x^+\). We call \(M\) the set of impulse, \(I\) the impulsive function. We define a function \(\phi : X \rightarrow R_+ \cup \infty\) as follows:

\[
\phi = \begin{cases} 
\infty, & \text{if } M^+(x) \notin \emptyset, \\
 s, & \text{if } \pi(x, t) M \text{ for } 0 < t < s, \pi(x, t) \in M,
\end{cases}
\]

Here we call \(s\) the time of absence of impulse of \(x\) such that \(s\) is the first time when \(\pi(x, 0)\) this \(M\).

Definition 2.2.3 A trajectory \(\pi_x\) is said to be periodic of period \(\tau\) and \(k\) if there exist positive integers \(m \geq 1\) and \(k \geq 1\) such that \(k\) is the smallest integer for which \(x^+_m = x^+_{m+k}\) and \(\tau_k = \sum_{i=m}^{m+k-1} \phi(x^+_i)\).

Theorem 2.1.1 (in [1]) Every continuous mapping of a closed bounded convex set in \(R^n\) into itself has a fixed point.

Consider the following general autonomous impulsive differential equation:

\[
\begin{align*}
\dot{x}(t) &= g_1(x, y), \\
\dot{y}(t) &= g_2(x, y), \\
\Delta x &= \sigma(x, y), \\
\Delta y &= \gamma(x, y),
\end{align*}
\]

(2.4)

Here \((x, y) \in R^2\) and \(g_1, g_2, \tau, y\) are all functions mapping \(R^2\) into \(R\), \(M \subset R^2\) is the set of impulse and we assume: (a1) \(g_1(x, y), g_2(x, y)\) are continuous with respect to \(x, y\) in \(R^2\); (b1) \(M \subset R^2\) is a line, \(\sigma(x, y)\) and \(\gamma(x, y)\) are linear functions of \(x\) and \(y\). For each point \(S(x, y) \in R^2\), we define \(I : R^2 \rightarrow R^2 : I(S) = (x^+, y^+) \in R^2, x^+ = x + \sigma(x, y), y^+ = y + \gamma(x, y)\). Obviously, \(N = I(M)\) is also a line of \(R^2\) or a subset of a line and we assume that \(N \cap M = \emptyset\). From Definition 2.2.2, we know that (2.4) is impulsive semi-dynamical system. The following theorem gives the Conditions on which (2.4) has a periodic solution of order one, defined by Definition 2.2.3.

Theorem 2.2.2 If system (2.4) satisfies assumptions (a1) and (b1) there
exists a bounded close simply connected region $D$ which has the following properties: (1) There is no singularity in it and the boundary $\partial D$ of satisfies 
$\{D - \partial D\} \cap M = \emptyset$. (2) $L_1 = D \cap M$ can not be tangent with trajectories of (2.4) except at end-points and $I(L_1) \subset D$. (3) Trajectories with initial point in $\partial D - L_1$ will enter into interior of $D$, then there must exist a periodic solution of system (2.4) of order one in region $D$.

3. Qualitative analysis of system (2.3) without impulsive effect

**Lemma 3.1.1** System (2.3) is uniformly bounded.

**Lemma 3.1.2** Suppose $\Gamma(t) = (S(t), x(t))$ is a periodic orbit of system (3.1) with $T$. The set $R$ consists of all the point in phase plane $\Gamma$.

Next, we will discuss the qualitative characteristic of system (2.3) without the impulsive effect. If no impulsive effective is introduced, The system (2.3) is

\[
\begin{aligned}
\dot{S}(t) &= Q(1 - S) - \frac{mS}{(a + bS)(c + S)} x, \\
\dot{x}(t) &= \frac{mS}{c + S} x - Qx.
\end{aligned}
\]  

We can obtain that (3.1) has one boundary equilibrium $A(1, 0)$ and one positive equilibrium $B(S^*, x^*)$ if $m > (1 + c)Q$, where $S^* = \frac{cQ}{m - Q} x^* = \frac{Q - cQ}{(a + bQ)(a + Q) + cQ}. \frac{Q - cQ}{(m - Q)^2}$. If $m \leq (1 + c)Q$, System (3.1) has only one boundary equilibrium $(1, 0)$ in region $R^+ = \{(S, x)|S \geq 0, x \geq 0\}$. Now we prove the equilibrium is globally asymptotically stable.

**Lemma 3.1.3** If $m < (1 + c)Q$, then the boundary equilibrium $A(1, 0)$ is a globally asymptotically stable. the boundary equilibrium $A$ is saddle-point if $m > (1 + c)Q$.

**Lemma 3.1.4** If $m > (1 + c)Q$, the positive equilibrium $B(S^*, x^*)$ is globally asymptotically stable node or focus and $\lim_{t \to \infty} S(t) = S^*$, $\lim_{t \to \infty} x(t) = x^*$. Where $S^* = \frac{cQ}{m - Q}, x^* = \frac{(Q - cQ)(am - Q) + dcQ}{(m - Q)^2} + \frac{mQ}{(m - Q)^2}$. From the Lemma 3.1.4, we have the following theorem:

**Theorem 3.1.4** the positive equilibrium $B(S^*, x^*)$ is globally asymptotically stable if $m > (1 + c)Q$.

4. Existence and stability of order one periodic solutions

4.1 Existence of order one periodic solution

We have known that $(S^*, x^*)$ is a stable node (or focus) when $m > (1 + c)Q$. The solution of (2.3) tend to the equilibrium $(S^*, x^*)$ and no impulsive control will occur when it satisfy $x(0) < x^*$ and $\frac{dx}{dt} \geq 0$ if $h \geq x^*$. So we mainly discuss the following case: $(H) h < x^*$ and $S(0) \leq 1$. 
We apply the existence criteria to prove order one periodic solution of system (2.3). Firstly, we need to construct a closed region so that the solution of enter the close region and retain there. We will illustrate there ideas by using Figs 1-2. From the Figs.1-2, we can obtain the line $x = h$ interacts the isocline line $\frac{mS}{c+S} = 0$, that is $\dot{x} = 0$, that $Q(1-S) - \frac{mS}{(a+bs)(c+S)}x = 0$ at the points $E(S_E, h)$ and $D(S_D, h)$, respectively. The impulsive set $m \subseteq \overline{EC}$, $\overline{EC} = \{(S, x)|S_E \leq S \leq 1\}$. The impulsive functions $I_1$ and $I_2$ map the impulsive set $M$ as $N = I(M) \subseteq \overline{FG}$, $\overline{FG} = \{(S, x)|(1-b)S_E \leq S \leq 1-b\}$, where $F(S_F, (1-b)h), G(S_G, (1-b)h), S_F = (1-b)S_E, S_G = (1-b)S_D$. We can have $S^+ = (1-b)S$ if $x = h$ and furthermore $S_F = (1-b)S_E \leq 1$. By the third equation of (2.3). According to the values of $S_F$ and $S_G$, we have the following cases: Case 1: $S_F \leq S_{FM}$ and $S_{FM} \leq S_G \leq S_{GM}$ (Fig.1) $S_{FM}$, $S_{GM}$ are the solution $\frac{mS}{c+S} = 0$, $Q(1-S) - \frac{mS}{(a+bs)(c+S)}x = 0$. Case 2: $S_F < S_G < S_{FM}$ (Fig.2).

**Theorem 4.1.1** If $m > (1+c)Q$, $0 < h < x^*$, $x_0 < h$ and $0 < S_0 < S_{GM}$, system (2.3) has an order one periodic solution.

**Proof**: Firstly, we know that the trajectories of system (2.3) starting from the region show in case 1 must interact with the segment $\overline{EC}$. Next, we construct the closed region $\Omega_1$. From the qualitative characteristic of system (3.1), we obtain that for $S = 1$, $\dot{x} = 0$ for $x = 0$ and $\dot{S} = 0$ for $S = 0$. The straight line $x = (1-b)h$ interacts $S = 0$ and $\frac{mS}{c+S} = 0$ at the points $H(0, (1-b)h)$ and $S_{FM}(S_{FM}, (1-b)h)$, respectively. Therefore, we have $\overline{AC}, \overline{CE}, \overline{ES_{FM}}, \overline{S_{FM}F}, \overline{FH}, \overline{HO}$ and $\overline{OA}$ (in Fig.1). We can see that the trajectories of system (2.3) enter and retain the region $\Omega_1$. By Theorem 2.2.2 we can obtain the system (2.3) have an order one periodic solution if the conditions of the theorem and case 1 hold.

Finally, we will prove that system has an order one periodic solution if the conditions of the theorem and case 2 hold. In Fig.2 we obtain the closed region consisting of $\overline{AC}, \overline{CE}, \overline{ES_{FM}}, \overline{S_{FM}G}, \overline{GH}, \overline{HO}$ and $\overline{OA}$ (in Fig.2). By Theorem 2.2.2 we can obtain the system (2.3) have an order one periodic solution if the conditions of the theorem and case 2 hold. To sum up, we have proved the Theorem 4.1.1.

**4.2 Stability of order one periodic solution**

**Lemma 4.2.1** The T-periodic solution $S = \xi(t), x = \eta(t)$ of the system

$$
\begin{align*}
\dot{S}(t) &= g_1(S, x), \\
\dot{x}(t) &= g_2(S, x), \\
\Delta S &= \sigma(S, x), \\
\Delta x &= \gamma(S, x),
\end{align*}
$$

is orbitally asymptotically stable if the Floquet Multiplier satisfies the condi-
\[ u_2 = \prod_{k=1}^{q} \Delta k \exp \left[ \int_0^T \frac{\partial g_1}{\partial S}(\xi(t), \eta(t)) + \frac{\partial g_2}{\partial S}(\xi(t), \eta(t)) dt \right], \]

\[ \Delta k = \frac{g_1(\frac{\partial x}{\partial S}, \frac{\partial \phi}{\partial S} - \frac{\partial \phi}{\partial S} + \frac{\partial \phi}{\partial S}) + g_2(\frac{\partial y}{\partial S} - \frac{\partial \phi}{\partial S} + \frac{\partial \phi}{\partial S})}{g_1 + g_2 \frac{\partial \phi}{\partial S}} \]

Where \( g_1, g_2, \frac{\partial g_1}{\partial S}, \frac{\partial g_2}{\partial S}, \frac{\partial \phi}{\partial S} \) and \( \frac{\partial \phi}{\partial S} \) are calculated at the point \((\xi(\tau_k), \eta(\tau_k))\), \( g_1^+ = g_1(\xi(\tau_k), \eta(\tau_k)), g_2^+ = g_2(\xi(\tau_k), \eta(\tau_k)). \phi(x, y) \) is a sufficiently smooth function with grad \( \phi(x, y) \neq 0 \) and \( \tau_k \in N \) is the time of kth jump. The Lemma is proved (\( \text{in} [5] \)).

Next, we suppose the periodic solution of system (4.1) with \( T \). By the points \( E_1^+((1-b)\xi, (1-b)h) \in \mathcal{FF} \) and \( (\xi, h) \in \mathcal{EE}(\mathcal{FF}, \mathcal{EC} \text{ as seen in Fig.1-2}) \). We analysis the stability of the positive periodic solution by Lemma 4.2.1. When the expression and the period of this solution are unknown, we can obtain:

\[ g_1(S, x) = Q(1-S) - \frac{mS}{c+S}x, g_2(S, x) = \frac{mS}{c+S}x - Qx, \sigma(S, x) = -bS, \]

\[ \gamma(S, x) = -bx, \phi(S, x) = x - h, \frac{\partial g_1}{\partial S} = -Q - \frac{mS}{c+S}, \frac{\partial g_2}{\partial S} = \frac{mS}{c+S} - Q, \]

\[ \frac{\partial \phi}{\partial S} = -b, \frac{\partial \phi}{\partial x} = 0, \frac{\partial \tau}{\partial S} = \frac{\partial \tau}{\partial x} = 0, \frac{\partial \phi}{\partial \tau} = 0, \frac{\partial \phi}{\partial \tau} = 1, (\xi(T^+), \eta(T^+)) = ((1-b)\xi, (1-b)h)_0. \Delta k = \]

\[ \frac{g_1(\frac{\partial x}{\partial S}, \frac{\partial \phi}{\partial S} - \frac{\partial \phi}{\partial S} + \frac{\partial \phi}{\partial S}) + g_2(\frac{\partial y}{\partial S} - \frac{\partial \phi}{\partial S} + \frac{\partial \phi}{\partial S})}{g_1 + g_2 \frac{\partial \phi}{\partial S}} = \frac{(1-b)^2}{2c+1}(Q \frac{mS}{c+S} - Q), \]

Denote \( G(t) = \frac{\partial g_1}{\partial S}(\xi(t), \eta(t)) + \frac{\partial g_2}{\partial S}(\xi(t), \eta(t)), then u_2 = \Delta k_1 \exp \left[ \int_0^T \frac{\partial g_1}{\partial S}(\xi(t), \eta(t)) + \frac{\partial g_2}{\partial S}(\xi(t), \eta(t)) dt \right] = \frac{(1-b)^2}{2c+1}(Q \frac{mS}{c+S} - Q) \times \exp \int_0^T G(t) dt. \) As \( (S(t), x(t)) \) is periodic solution of system (4.1), is proved in Lemma 3.1.2 so that we have \( \exp \int_0^T G(t) dt. \) therefore \( |u_2| \leq 1 \) if \( |(1-b)^2 \frac{mS}{c+1}(Q \frac{mS}{c+1} - Q) - mS \frac{mS}{c+1} - Q| < 1 \) holds.

**Theorem 4.2.1** System (4.1) with conditions of Theorem 4.1.1 have an order one periodic solution. Furthermore, this order one periodic solution is stable if \( |(1-b)^2 \frac{mS}{c+1}(Q \frac{mS}{c+1} - Q) - mS \frac{mS}{c+1} - Q| < 1 \) holds.
References


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