Some Invariant Subsets of $Q^*(\sqrt{n})$

under the Action of $PSL(2, Z)$

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Abstract

In this paper, we study the action of the Modular Group $G = \langle x, y : x^2 = y^3 = 1 \rangle$ on $Q^*(\sqrt{n})$ when $n \not\equiv 0 \pmod{2p}$. We find that there exist two $G$-subsets of $Q^*(\sqrt{n})$ if $n$ is a quadratic residue or quadratic non-residue modulo $2p$ and there exist four $G$-subsets of $Q^*(\sqrt{n})$ when $n \equiv p \pmod{2p}$.

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1 Introduction and Preliminaries

We denote $\alpha = \frac{a + \sqrt{n}}{c}$ by $\alpha(a, b, c)$. Let $u > 1$ be a fixed integer. We say that two classes $\alpha(a, b, c)$ and $\alpha'(a', b', c')$ of $Q^*(\sqrt{n})$ are $u$-equivalent (and write $\alpha(a, b, c) \sim_u \alpha'(a', b', c')$ or $\alpha \sim_u \alpha'$) if and only if $a \equiv a' \pmod{u}$, $b \equiv b' \pmod{u}$ and $c \equiv c' \pmod{u}$. A set $X$ with an action of some group $G$ on it, is known as a $G$-set. Let $n = k^2m$, where $k$ is any integer and $m$ is square free positive integer. Then the set $Q^*(\sqrt{n}) = \{ \frac{a + \sqrt{n}}{c} : a, b = \frac{a^2-n}{c}, c \in \}$
$Z$ and $(a, b, c) = 1$ is a $G$-subset of $Q(\sqrt{m})$ under the action of the modular group $G = \langle x, y : x^2 = y^3 = 1 \rangle$ where $x(\alpha) = \frac{1}{\alpha}$, $y(\alpha) = \frac{\alpha - 1}{\alpha}$ are the linear fractional transformations. It is well known that every real quadratic irrational number $u + v\sqrt{m}$ of $Q(\sqrt{m})$ can be written uniquely as $a + \sqrt{n}c$, where $n$ is a non-square positive integer and $(a, \frac{a^2-n}{c}, c) = 1$. Action of the modular group $G = \langle x, y : x^2 = y^3 = 1 \rangle$ on certain subsets of the real quadratic field $Q(\sqrt{m})$ has been discussed earlier in [1, 2, 4] and [5]. In a recent work, M. Aslam Malik and M. Asim Zafar [1], have discussed the $G$-subsets of $Q^*(\sqrt{n})$ when $n \equiv 0(\mod 2p)$. Thus it becomes interesting to know the $G$-subsets of $Q^*(\sqrt{n})$ when $n \not\equiv 0(\mod 2p)$. In this paper we investigate the $G$-subsets of $Q^*(\sqrt{n})$ when $n$ is a quadratic residue of $2p$ or quadratic non-residue of $2p$ and show that in either case there exist two proper $G$-subsets. Further we prove that there exist four $G$-subsets of $Q^*(\sqrt{n})$ when $n \equiv p(\mod 2p)$. Notations used in this paper are standard and we follow [5] and [6]. In particular, $\left(\frac{a}{p}\right)$ denotes the Legendre of $a$ modulo $p$. We need the following results of [6], for use in the sequel.

**Theorem 1.1** Let $p$ be an odd prime. Then

1. $\left(\frac{a}{p}\right) \equiv a^{p-1/2} \pmod{p}$
2. $\left(\frac{-1}{p}\right) = (-1)^{p-1/2}$

**Theorem 1.2** Let $m$ be a positive integer with canonical decomposition $2^{e_0} \prod p_i^{e_i}$ and $a$ any integer with $(a, m) = 1$. Then $x^2 \equiv a \pmod{m}$ is solvable if and only if $x^2 \equiv a \pmod{2^{e_0}}$ and $x^2 \equiv a \pmod{p_i^{e_i}}$ are solvable.

**Theorem 1.3**

1. Let $p$ be any odd prime such that $p \equiv 1(\mod 4)$ and $a$ be a quadratic residue of $p$ then $p - a$ is a quadratic residue of $p$.
2. Let $p$ be any odd prime such that $p \equiv 3(\mod 4)$ and $a$ be a quadratic residue of $p$ then $p - a$ is a quadratic non-residue of $p$. 
2 Action of $G$ on $Q^*(\sqrt{n})$ when $\left(\frac{n}{p}\right) = 1$ or $-1$

We know that,

$$x\left(\frac{a + \sqrt{n}}{c}\right) = \frac{-a + \sqrt{n}}{b}$$  \hspace{1cm} (1)

$$y\left(\frac{a + \sqrt{n}}{c}\right) = \frac{(-a + b) + \sqrt{n}}{b}$$  \hspace{1cm} (2)

To find the $G$-subsets of $Q^*(\sqrt{n})$ when $n$ is a quadratic residue or quadratic non-residue of $2p$, the following two lemmas are needed.

**Lemma 2.1** Let $\alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n})$ and $p \equiv 3(mod\ 4)$ be a prime divisor of $a$. Then $\left(\frac{b}{p}\right) = -\left(\frac{c}{p}\right)$ or $\left(\frac{b}{p}\right) = \left(\frac{c}{p}\right)$ according as $n$ is a quadratic residue or quadratic non-residue of $p$.

Proof. Let $n$ be a quadratic residue of $2p$ such that $p \mid a$. Then $x^2 \equiv n(mod\ 2p)$ is solvable. So by Theorem 1.2, $x^2 \equiv n(mod\ p)$ is solvable and hence $\left(\frac{a}{p}\right) = 1$. Also $p \mid a$, so the relation $a^2 - n = bc$ implies $0^2 - n \equiv bc(mod\ p)$. That is,

$$p - n \equiv bc(mod\ p).$$  \hspace{1cm} (3)

Since $p \equiv 3(mod\ 4)$ so by Theorem 1.3, $p - n$ is a quadratic non-residue of $p$. Hence by (3), $bc$ is a quadratic non-residue of $p$. Therefore one of $b$ or $c$ is a quadratic residue of $p$ and other is a quadratic non-residue of $p$. Consequently, $\left(\frac{b}{p}\right) = -\left(\frac{c}{p}\right)$.

Similarly if $n$ is a quadratic non-residue of $2p$ such that $p \mid a$, so again by Theorem 1.2, $x^2 \equiv n(mod\ p)$ is not solvable since $x^2 \equiv n(mod\ 2)$ always admits a solution. This means that $\left(\frac{a}{p}\right) = -1$. Also $a \equiv 0(mod\ p)$. So the relation $a^2 - n = bc$ gives $0^2 - n \equiv bc(mod\ p)$. Then,

$$n \equiv -bc(mod\ p)$$  \hspace{1cm} (4)

As $n$ is a quadratic non-residue of $p$, so by (4), we have,

$$-1 = \left(\frac{n}{p}\right) = \left(\frac{-bc}{p}\right) = \left(\frac{-1}{p}\right) \left(\frac{b}{p}\right) \left(\frac{c}{p}\right)$$  \hspace{1cm} (5)
Since \( p \equiv 3(\text{mod } 4) \), so by Theorem 1.1, (5) implies that
\[
\left(\frac{b}{p}\right) = 1
\]
Which clearly shows that both \( b \) and \( c \) are quadratic residues of \( 2p \) or quadratic
non-residues of \( p \). Hence, \( \left(\frac{b}{p}\right) = \left(\frac{c}{p}\right) \). \( \square \)

The proof of the following lemma is analogous to the proof of lemma 2.1.

**Lemma 2.2** Let \( \alpha = a + \sqrt{n}c \in Q^*(\sqrt{n}) \) and \( p \equiv 1(\text{mod } 4) \) be a prime divisor of \( a \). Then  \( \left(\frac{b}{p}\right) = \left(\frac{c}{p}\right) \) or \( \left(\frac{b}{p}\right) = -\left(\frac{c}{p}\right) \) according as \( n \) is a quadratic residue or quadratic non-residue of \( 2p \).

The following lemma of [4] is of crucial importance.

**Lemma 2.3** Let \( \alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n}) \), with \( b = \frac{a^2 - n}{c} \) and \( (a, b, c) = 1 \). Then the sets
\[
A = \{ \alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n}) : 2 \mid (b, c) \} \quad \text{and} \quad B = \{ \alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n}) : 2 \nmid (b, c) \}
\]
are both \( G \)-subsets of \( Q^*(\sqrt{n}) \).

To explore the \( G \)-subsets of \( Q^*(\sqrt{n}) \) when \( n \) is a quadratic residue or quadratic non-residue of \( 2p \), we discuss the case when \( p \equiv 3(\text{mod } 4) \) in the following theorem and the case when \( p \equiv 1(\text{mod } 4) \) can be proved in a similar technique.

**Theorem 2.4** Let \( \alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n}) \) and \( p \) be a prime such that \( p \equiv 3(\text{mod } 4) \). Then.
(a) If \( n \) is a quadratic residue of \( 2p \), then the sets
\[
A_1 = \{ \alpha \in A : \left(\frac{a}{p}\right) = \pm 1 \text{ or if } p \mid a \text{ then } \left(\frac{b}{p}\right) = -\left(\frac{c}{p}\right) \}
\]
\[
B_1 = \{ \alpha \in B : \left(\frac{a}{p}\right) = \pm 1 \text{ or if } p \mid a \text{ then } \left(\frac{b}{p}\right) = -\left(\frac{c}{p}\right) \}
\]
are the \( G \)-subsets of \( Q^*(\sqrt{n}) \).
(b) If \( n \) is a quadratic non-residue of \( 2p \), then the sets
\[ A_2 = \{ \alpha \in A : \left( \frac{\alpha}{p} \right) = \pm 1 \text{ or if } p \mid a \text{ then } \left( \frac{b}{p} \right) = \left( \frac{\alpha}{p} \right) \} \]

\[ B_2 = \{ \alpha \in B : \left( \frac{\alpha}{p} \right) = \pm 1 \text{ or if } p \mid a \text{ then } \left( \frac{b}{p} \right) = \left( \frac{\alpha}{p} \right) \} \]

are the \( G \)-subsets of \( Q^*(\sqrt{n}) \)

Proof. Let \( \alpha = \frac{a + \sqrt{n}}{c} \in A_1 \) with \( n \) is a quadratic residue of \( 2p \). Since each \( g \in G \) can be expressed in terms of \( x, y \). Therefore, it is sufficient to show that the action of \( x, y \) on \( \alpha \) belongs to \( A_1 \). Since \( n \) is a quadratic residue of \( 2p \), so by Theorem 1.3, \( n \) is a quadratic residue of \( p \). First, we see that \( A_1 \) is invariant under \( x \). For this, we know, \( x(\alpha) = \frac{-a + \sqrt{n}}{b} = \frac{a_1 + \sqrt{n}}{c_1} \), where, \( a_1 = -a \), \( b_1 = c \), \( c_1 = b \). Let \( p \mid a \) with \( \left( \frac{b}{p} \right) = -\left( \frac{c}{p} \right) \). Then \( a_1 = -a \equiv 0 \pmod{p} \). Also under the action of \( x, b \) and \( c \) have the same significance instead of replacement of each other. We, therefore, have \( \left( \frac{b_1}{p} \right) = -\left( \frac{c_1}{p} \right) \). Similarly, if \( \left( \frac{b}{p} \right) = \pm 1 \) then \( a_1 = -a \not\equiv 0 \pmod{p} \). Since there always exist \( \frac{p-1}{2} \) quadratic residues and \( \frac{p-1}{2} \) quadratic non-residues of an odd prime \( p \). So, \( \left( \frac{a}{p} \right) = \pm 1 \).

Hence in either case \( x(\alpha) \in A_1 \). To see that \( A_1 \) is invariant under \( y \). We take \( y(\frac{a + \sqrt{n}}{c}) = \frac{-a + b + \sqrt{n}}{b} = \frac{a_2 + \sqrt{n}}{c_2} \), where, \( a_2 = -a + b \), \( b_2 = -2a + b + c \), and \( c_2 = b \).

Let \( p \mid a \) with \( \left( \frac{b}{p} \right) = -\left( \frac{c}{p} \right) \). Then, \( a_2 \equiv b \pmod{p} \) but \( \left( \frac{b}{p} \right) = -\left( \frac{c}{p} \right) \). Hence \( a_2 \) is either a quadratic residue or quadratic non-residue of \( p \). This implies that \( \left( \frac{a_2}{p} \right) = \pm 1 \) and therefore \( y(\alpha) \in A_1 \). Finally, if \( \left( \frac{a}{p} \right) = \pm 1 \) then \( a_2 = -a + b \equiv 0 \pmod{p} \) or \( a_2 = -a + b \not\equiv 0 \pmod{p} \). Suppose \( a_2 = -a + b \equiv 0 \pmod{p} \) then the equation \( a_2^2 - n = b_2c_2 \) gives \( -n = (-2a + b + c)c \equiv 0 \pmod{p} \). Since \( n \) is a quadratic residue of \( p \), so by Lemma 2.1, we have \( \left( \frac{-2a + b + c}{p} \right) = -\left( \frac{c}{p} \right) \)

or \( \left( \frac{a_2}{p} \right) = -\left( \frac{a}{p} \right) \). If \( a_2 = -a + b \not\equiv 0 \pmod{p} \) then as explained above, we have, \( \left( \frac{a_2}{p} \right) = \pm 1 \). Therefore in either case \( y(\alpha) \in A_1 \). Consequently, the set \( A_1 \) is mapped onto itself under \( x \) and \( y \) and, therefore, it is a \( G \)-subset of \( Q^*(\sqrt{n}) \). Similarly the sets \( B_1, A_2 \) and \( B_2 \) are the \( G \)-subsets of \( Q^*(\sqrt{n}) \).

\[ \square \]

**Theorem 2.5** Let \( \alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n}) \) and \( p \) be a prime such that \( p \equiv 1 \pmod{4} \), Then.

(a) If \( n \) is a quadratic residue of \( 2p \), then the sets
\[ A_1 = \{ \alpha \in A : \left( \frac{\alpha}{p} \right) = \pm 1 \text{ or if } p \mid a \text{ then } \left( \frac{b}{p} \right) = \left( \frac{c}{p} \right) \} \]

\[ B_1 = \{ \alpha \in B : \left( \frac{\alpha}{p} \right) = \pm 1 \text{ or if } p \mid a \text{ then } \left( \frac{b}{p} \right) = \left( \frac{c}{p} \right) \} \]

are the $G$-subsets of $Q^*(\sqrt{n})$

(b) If $n$ is a quadratic non-residue of $2p$, then the sets

\[ A_2 = \{ \alpha \in A : \left( \frac{\alpha}{p} \right) = \pm 1 \text{ or if } p \mid a \text{ then } \left( \frac{b}{p} \right) = -\left( \frac{c}{p} \right) \} \]

\[ B_2 = \{ \alpha \in B : \left( \frac{\alpha}{p} \right) = \pm 1 \text{ or if } p \mid a \text{ then } \left( \frac{b}{p} \right) = -\left( \frac{c}{p} \right) \} \]

are the $G$-subsets of $Q^*(\sqrt{n})$

**Example 2.5** Take $n \equiv 1 (mod \ 6)$. Since the set \{0, 1, 2, 3, 4, 5\} has only one quadratic residue of 6 namely 1, so by Theorem 2.4, the set $A_1$ contains classes of the form, \{0, 1, 5\}, \{1, 0, 1\}, \{1, 0, 3\}, \{1, 0, 5\}, \{1, 1, 0\}, \{1, 2, 3\}, \{1, 3, 0\}, \{1, 3, 2\}, \{1, 3, 4\}, \{1, 4, 3\}, \{1, 5, 0\}, \{3, 1, 2\}, \{3, 4, 5\}, \{4, 1, 3\}, \{4, 3, 1\}, \{4, 3, 3\}, \{4, 3, 5\}, \{4, 5, 3\}, \{0, 5, 1\}, \{5, 1, 0\}, \{5, 3, 0\}, \{5, 5, 0\}, \{5, 0, 1\}, \{5, 3, 2\}, \{5, 0, 3\}, \{5, 2, 3\}, \{5, 4, 3\}, \{5, 3, 4\}, \{5, 0, 5\}, \{3, 2, 1\}, \{3, 5, 4\}, \{2, 3, 1\}, \{2, 1, 3\}, \{2, 3, 3\}, \{2, 5, 3\} and \{2 3 5\}

Then by equations (1) and (2), it can be seen that $A_1$ is mapped onto itself under the transformations $x$ and $y$. Hence it is a $G$-subset of $Q^*(\sqrt{n})$. Similarly, we can find other $G$-subsets of $Q^*(\sqrt{n})$.

### 3 Action of $G$ on $Q^*(\sqrt{n})$ when $n \equiv p(mod \ 2p)$

In the following theorem, we show that there exist four $G$-subsets of $Q^*(\sqrt{n})$ when $n \equiv p(mod \ 2p)$

**Theorem 3.1** Let $p$ be an odd prime such that $n \equiv p(mod \ 2p)$ and

\[ \alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}), \text{ with } b = \frac{a^2-n}{c}. \]

Then the following sets.

$A_1 = \{ \alpha \in A : \left( \frac{b}{p} \right) = \left( \frac{c}{p} \right) = 1 \text{ or if } p \text{ divides one of } b \text{ or } c \text{ then other is a} \]
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quadratic residue of $p$ }

\[ A_2 = \{ \alpha \in A : \left( \frac{\alpha}{p} \right) = \left( \frac{b}{p} \right) = -1 \text{ or if } p \text{ divides one of } b \text{ or } c \text{ then other is a quadratic residue of } p \} \]

\[ B_1 = \{ \alpha \in B : \left( \frac{\alpha}{p} \right) = \left( \frac{a}{p} \right) = 1 \text{ or if } p \text{ divides one of } b \text{ or } c \text{ then other is a quadratic residue of } p \} \]

\[ B_2 = \{ \alpha \in B : \left( \frac{\alpha}{p} \right) = \left( \frac{c}{p} \right) = -1 \text{ or if } p \text{ divides one of } b \text{ or } c \text{ then other is a quadratic non-residue of } p \} \]

are the $G$-subsets of $\mathbb{Q}^*(\sqrt{n})$.

Proof. Let $\alpha = \frac{a + \sqrt{n}}{c} \in A_1$. We show that the action of $x, y$ on $\alpha$ belongs to $A_1$. To show that $A_1$ is invariant under $x$. We know, $x(\alpha) = \frac{-a + \sqrt{n}}{b} = \frac{a_1 + \sqrt{n}}{c_1}$, where, $a_1 = -a$, $b_1 = c$, $c_1 = b$. Let $\left( \frac{b}{p} \right) = \left( \frac{c}{p} \right) = 1$. If $a \equiv 0(mod \ p)$, then $a_1 = -a \equiv 0(mod \ p)$. Therefore, the equation $b_1c_1 = a_1^2 - n$ reduces to $b_1c_1 \equiv 0(mod \ p)$. This shows that one of $b$ or $c$ is divisible by $p$ which is a contradiction. Thus if $b$ and $c$ both are quadratic residues of $p$ then $a \equiv 0(mod \ p)$ is not possible. We, therefore, suppose $a \not\equiv 0(mod \ p)$ then $a_1 \not\equiv 0(mod \ p)$ which implies that $a_1^2$ is a quadratic residue of $p$. Then $b_1c_1 \equiv a_1^2(mod \ p)$ gives that both $b_1$ and $c_1$ are the quadratic residues of $p$. Next, we suppose, $p$ divides one of $b$ or $c$ and other is a quadratic residue of $p$. Again if $a \not\equiv 0(mod \ p)$ then $a_1 = -a \not\equiv 0(mod \ p)$. So the relation $b_1c_1 \equiv a_1^2(mod \ p)$ implies that both $b_1$ and $c_1$ are either quadratic residues or quadratic non-residues of $p$ which is a contradiction. Thus we take $a \equiv 0(mod \ p)$. Then we must have $b_1c_1 \equiv 0(mod \ p)$. This certainly implies that one of them must be divisible by $p$ and other is a quadratic residue of $p$. Hence in either case $x(\alpha) \in A_1$. To see that $A_1$ is invariant under $y$. We take $y(\frac{a + \sqrt{n}}{c}) = \frac{-a + b + \sqrt{n}}{b} = \frac{a_2 + \sqrt{n}}{c_2}$, where, $a_2 = -a + b$, $b_2 = -2a + b + c$, and $c_2 = b$. Let $\left( \frac{b}{p} \right) = \left( \frac{c}{p} \right) = 1$ then $a \not\equiv 0(mod \ p)$. So $b_2c_2 = a_2^2 - n$ yields that $b_2c_2 \equiv a_2^2(mod \ p)$. Now, if $a_2 = -a + b \not\equiv 0(mod \ p)$ then $b_2c_2$ is a quadratic residue of $p$. Since $c_2 = b$ is a quadratic residue of $p$ so is $b_2$ as well. But if $a_2 = -a + b \equiv 0(mod \ p)$ then we have, $b_2c_2 \equiv 0(mod \ p)$. Thus $p$ must divide $b_2$ and $c_2 = b$ is a quadratic residue of $p$. Thus in either case $y(\alpha) \in A_1$. Finally, we suppose $p$ divides one of $b$ or $c$ and other is a quadratic residue of $p$. Then $p$ divides $a$. Hence $a_2 = -a + b \equiv b(mod \ p)$. Let $p$ divides $b$ with $c$ as the quadratic residue. Then $b_2 = -2a + b + c \equiv c(mod \ p)$ and $c_2 = b \equiv 0(mod \ p)$. This shows that $p$ divides $c_2$ with $b_2$ as the quadratic residue of $p$. Consequently, $A_1$ is invariant under
the action of \( x \) and \( y \). Similarly, we can show that the sets \( A_2 \), \( B_1 \) and \( B_2 \) are the \( G \)-subsets.

We conclude this paper with the following remarks.

**Remarks 3.2**

(1) If \( n \equiv 0 \,(\text{mod } pq) \), where \( p \) and \( q \) are distinct odd primes. then \( Q^*(\sqrt{n}) \) contains four \( G \)-subsets. As we know by Theorem 3.1, that if \( n \equiv p \,(\text{mod } 2p) \), then \( Q^*(\sqrt{n}) \) has four \( G \)-subsets. Since \( n \equiv p \,(\text{mod } 2p) \), so there exist some integer \( t \) such that \( n = p + 2pt = p(1 + 2t) = pt_1 \), where \( t_1 = 1 + 2t \), now we choose the values of \( t \) for which \( t_1 \) is an odd prime \( q \) different from \( p \), then we have \( n \equiv 0 \,(\text{mod } pq) \).

(2) If \( n \) is a quadratic residue of \( 2p \), then the \( G \)-subsets of \( Q^*(\sqrt{n}) \) namely \( A_1 \) and \( B_1 \) mentioned in Theorem 2.4, contain \( 3p(p+1) \) and \( p(p+1) \) classes respectively.

(3) If \( n \) is a quadratic non-residue of \( 2p \), then the \( G \)-subsets of \( Q^*(\sqrt{n}) \) namely \( A_2 \) and \( B_2 \) mentioned in Theorem 2.4, contain \( 3p(p-1) \) and \( p(p-1) \) classes respectively.

(4) If \( n \equiv p \,(\text{mod } 2p) \), then two of the four \( G \)-subsets of \( Q^*(\sqrt{n}) \) namely \( A_1 \) and \( A_2 \) mentioned in Theorem 3.1, contain \( \frac{3}{2}(p^2 - 1) \) classes and the other two namely \( B_1 \) and \( B_2 \) mentioned in Theorem 3.1, contain \( \frac{p^2-1}{2} \) classes respectively.

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