A Note on Relative $L$-Order and Relative $L^*$-Order of Entire Functions

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Abstract

In the paper we consider regular $L$ relative growth and regular $L'$ relative growth of an entire function with respect to another entire function and show that under certain conditions they are equal where $L \equiv L(r)$ is a slowly changing function.

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1 Introduction, Definitions and Notations.

Let \( f \) and \( g \) be two entire functions and

\[
F(r) = \max\{|f(z)| : |z| = r\} \quad \text{and} \quad G(r) = \max\{|g(z)| : |z| = r\}.
\]

If \( f \) is non-constant then \( F(r) \) is strictly increasing and continuous and its inverse \( F^{-1} : (|f(0)|, \infty) \to (0, \infty) \) exists and is such that

\[
\lim_{s \to \infty} F^{-1}(s) = \infty.
\]

The order \( \rho_f \) of \( f \) \{cf. [2], [4]\} is given by

\[
\rho_f = \inf \{\mu > 0 : F(r) < \exp(r^\mu) \text{ for all } r > r_0(\mu) > 0\}
\]

\[
= \limsup_{r \to \infty} \frac{\log \log F(r)}{\log r}
\]

where \( \log^{[k]} x = \log (\log^{[k-1]} x) \) for \( k = 1, 2, 3, \ldots \) and \( \log^{[0]} x = x \). The lower order of \( f \) denoted by \( \lambda_f \) is defined as

\[
\lambda_f = \liminf_{r \to \infty} \frac{\log \log F(r)}{\log r}.
\]

If \( \rho_f \) and \( \lambda_f \) are equal then \( f \) is said to be of regular growth \{cf. [5]\}. Recently Bernal [1] introduced the idea of relative order of \( f \) with respect to \( g \), denoted by \( \rho_g(f) \), as follows:

\[
\rho_g(f) = \inf \{\mu > 0 : F(r) < G(r^\mu) \text{ for all } r > r_0(\mu) > 0\}
\]

\[
= \limsup_{r \to \infty} \frac{\log G^{-1}F(r)}{\log r}
\]

where \( g \) is a non-constant entire function \( g \). The definition coincides with the classical one if \( g(z) = \exp z \). If \( f \) is non-constant and \( g = f \) then \( \rho_g(f) = 1 \). As in the classical case we define the relative lower order of \( f \) with respect to a non-constant entire function \( g \) denoted by \( \lambda_g(f) \) as follows:

\[
\lambda_g(f) = \liminf_{r \to \infty} \frac{\log G^{-1}F(r)}{\log r}.
\]

If \( \rho_g(f) = \lambda_g(f) \) then \( f \) is said to be of regular relative growth with respect to \( g \). Therefore if \( f \) is of regular relative growth with respect to a non-constant entire function \( g \), we have

\[
\rho_g(f) = \lim_{r \to \infty} \frac{\log G^{-1}F(r)}{\log r}.
\]
Clearly if \( f \) is of regular relative growth with respect to \( g(z) = \exp z \) then \( f \) is also of regular growth. Somasundaram and Thamizharasi [3] introduced the notions of \( L \)-order and \( L^* \)-order for entire functions where \( L \equiv L(r) \) is a positive continuous function increasing slowly i.e., \( L(ar) \sim L(r) \) as \( r \to \infty \) for every positive constant ‘\( a \)’. With the help of the above notion we may define relative \( L \)-order and relative \( L^* \)-order. The following definitions are then obvious.

**Definition 1** [3] The \( L \)-order \( \rho_f^L \) and the \( L \)-lower order \( \lambda_f^L \) of an entire function \( f \) are defined as follows:

\[
\rho_f^L = \limsup_{r \to \infty} \frac{\log^2 M(r, f)}{\log[rL(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r \to \infty} \frac{\log^2 M(r, f)}{\log[rL(r)]}.
\]

**Definition 2** [3] The \( L^* \)-order \( \rho_f^{L^*} \) and \( L^* \)-lower order \( \lambda_f^{L^*} \) of an entire function \( f \) are defined as follows:

\[
\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log M(r, f)}{[rL^*(r)]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log M(r, f)}{[rL^*(r)]}.
\]

**Definition 3** The relative \( L \)-order and relative lower \( L \)-order of an entire function \( f \) with respect to an entire function \( g \) respectively denoted by \( \rho_g^L(f) \) and \( \lambda_g^L(f) \) are defined as

\[
\rho_g^L(f) = \limsup_{r \to \infty} \frac{\log G^{-1}F(r)}{\log[rL(r)]} \quad \text{and} \quad \lambda_g^L(f) = \liminf_{r \to \infty} \frac{\log G^{-1}F(r)}{\log[rL(r)]}.
\]

**Definition 4** The relative \( L^* \)-order and relative lower \( L^* \)-order of an entire function \( f \) with respect to an entire function \( g \) with rate \( t \) respectively denoted by \( \rho_g^{L^*}(f) \) and \( \lambda_g^{L^*}(f) \) are defined as

\[
\rho_g^{L^*}(f) = \limsup_{r \to \infty} \frac{\log G^{-1}F(r)}{\log[rL^*(r)]} \quad \text{and} \quad \lambda_g^{L^*}(f) = \liminf_{r \to \infty} \frac{\log G^{-1}F(r)}{\log[rL^*(r)]}.
\]

In fact Definition 2 and Definition 4 are more generalised than Definition 1 and Definition 3 respectively. The following definition is most generalised.

**Definition 5** The relative \( L^* \)-order and relative lower \( L^* \)-order of an entire function \( f \) with respect to an entire function \( g \) with rate \( t \) respectively denoted by \( ^{(t)}\rho_g^{L^*}(f) \) and \( ^{(t)}\lambda_g^{L^*}(f) \) are defined as

\[
^{(t)}\rho_g^{L^*}(f) = \limsup_{r \to \infty} \frac{\log G^{-1}F(r)}{\log[r \exp^[t]L(r)]} \quad \text{and} \quad ^{(t)}\lambda_g^{L^*}(f) = \liminf_{r \to \infty} \frac{\log G^{-1}F(r)}{\log[r \exp^[t]L(r)]}.
\]
where $t = 1, 2, 3, ...$

In the paper we prove a few theorems on the relationship between $\rho^L_g(f)$ and $\rho^L_f$. We do not explain the standard notations and definitions in the theory of entire functions as those are available in [5]. Throughout the paper we assume $f, g$ etc. as non-constant functions, unless otherwise stated.

2 Theorems.

In this section we present the main results of the paper.

**Theorem 1** If $f$ be the $L$-regular growth and of $L$-regular relative growth with respect to $g$ and $\rho^L_g(f) = \rho^L_f > 0$ then $g$ is of $L$-regular growth of $L$-order one. Conversely if $g$ is of $L$-regular growth of order one then $\rho^L_g(f) = \rho^L_f$ for every entire $f$ with $L$-regular relative growth.

**Proof.** Let us first suppose that $\rho^L_g(f) = \rho^L_f(f) = \rho > 0$.

Also let $0 < \varepsilon < 1$. Let us set $\varepsilon_1 = \frac{\rho \varepsilon}{2 + \varepsilon}$.

So $\varepsilon_1 < \rho$.

Then there exists $r_0 > 0$ such that for $r \geq r_0$

$$F(r) < \exp[(rL(r))^{\rho + \varepsilon_1}] \text{ and } F(r) > \exp[(rL(r))^{\rho - \varepsilon_1}]. \quad (1)$$

Also $F(r) < G(r^{\rho + \varepsilon_1})$ and $F(r) > G(r^{\rho - \varepsilon_1})$. \quad (2)

From (1) and (2) we get for $r \geq r_0$

$$G(r^{\rho - \varepsilon_1}) < F(r) < \exp[(rL(r))^{\rho + \varepsilon_1}]$$

and therefore for $r \geq r_0^\rho$ we obtain from above that

$$G(r) < \exp[(rL(r))^{\frac{\rho + \varepsilon_1}{\rho - \varepsilon_1}}] = \exp[(rL(r))^{1 + \frac{2\varepsilon_1}{\rho - \varepsilon_1}}].$$

So, $G(r) < \exp[(rL(r))^{1 + \varepsilon}]$ for $r \geq r_0^\rho$. \quad (3)
Similarly from (1) and (2) we obtain that
\[
\exp\left[\{(rL(r))^{1-\varepsilon}\}^2\right] < G(r) \text{ for } r \geq r_0^{2\rho}. \tag{4}
\]
So from (3) and (4), for \(r \geq r_0^{2\rho}\)
\[
\exp\left[\{(rL(r))^{1-\varepsilon}\}^2\right] < G(r) < \exp\left[\{(rL(r))^{1+\varepsilon}\}^2\right].
\tag{5}
\]
So \(g(z)\) is of \(L\)-regular growth of order one. Conversely for \(\varepsilon > 0\) there exists \(r_1 > 0\) such that for \(r \geq r_1\),
\[
\exp\left[\{(rL(r))^{1-\varepsilon}\}^2\right] < G(r) < \exp\left[\{(rL(r))^{1+\varepsilon}\}^2\right].
\tag{5}
\]
Also from the definition of \(\rho^L_g(f)\), there exists \(r_2 > 0\) such that for \(r \geq r_2\),
\[
G(r^{\rho^L(f)}) < F(r) < G(r^{\rho^L(f)+\varepsilon}). \tag{6}
\]
From (5) and (6), we have for \(r \geq r_3 = \max(r_1, r_2)\),
\[
\exp\left[\{rL(r)\}^{\rho^L(f)\varepsilon(1+\rho^L(f)+\varepsilon)}\right] < F(r) < \exp\left[\{rL(r)\}^{\rho^L(f)+\varepsilon(1+\rho^L(f)+\varepsilon)}\right]. \tag{7}
\]
Since \(\varepsilon > 0\) is arbitrary, from (7) we obtain that
\[
\rho^L_f = \lim_{r \to \infty} \frac{\log \log F(r)}{\log \{rL(r)\}} = \rho^L_g(f).
\]
This proves the theorem. In the next theorem we see the more generalisation of Theorem 1. ■

**Theorem 2** If \(f\) be of \(L^*\)-regular growth and \(L^*\)-regular relative growth with respect to \(g\) and \(\rho^L_g(f) = \rho^L_f > 0\) then \(g\) is of \(L^*\)-regular growth of \(L^*\)-order one. Conversely if \(g\) is of \(L^*\)-regular growth of \(L^*\)-order one then \(\rho^L_g(f) = \rho^L_f\) for every entire \(f\) with \(L^*\)-regular relative growth.

**Proof.** Let us forst suppose that
\[
\rho^L_g(f) = \rho^L_f = \rho > 0.
\]
Also let \(0 < \varepsilon < 1\).

Let us set \(\varepsilon_1 = \frac{\rho \varepsilon}{2 + \varepsilon}\)
So \(\varepsilon_1 < \rho\).
Then there exists $r_0 > 0$ such that for $r \geq r_0$
\[
F(r) < \exp\{re^{L(r)}\rho^{+\varepsilon_1}\} \quad \text{and} \quad F(r) > \exp\{re^{L(r)}\rho^{-\varepsilon_1}\}.
\]  
(8)

From (2) and (8) for $r \geq r_0$,
\[
G(r^{\rho^{+\varepsilon_1}}) < F(r) < \exp\{re^{L(r)}\rho^{+\varepsilon_1}\}
\]
and therefore for $r \geq r_0^\rho$ we obtain from above that
\[
G(r) < \exp\{re^{L(r)}\rho^{+\varepsilon_1}\} = \exp\{re^{L(r)}\frac{2\varepsilon_1}{\rho^{+\varepsilon_1}}\}.
\]
So, $G(r) < \exp\{\{re^{L(r)}\}^{1+\varepsilon}\}$ for $r \geq r_0^\rho$.

(9)

Similarly from (2) and (8) we obtain that
\[
\exp\{\{re^{L(r)}\}^{1-\varepsilon}\} < G(r) \quad \text{for} \quad r \geq r_0^{2\rho}.
\]

(10)

So $g$ is of $L^*$-regular growth of order one. Conversely for $\varepsilon > 0$ there exists $r_1 > 0$ such that for $r \geq r_1$,
\[
\exp\{\{re^{L(r)}\}^{1-\varepsilon}\} < G(r) < \exp\{\{re^{L(r)}\}^{1+\varepsilon}\}.
\]

(11)

Also from the definition of $\rho^*_g(f)$, there exists $r_2 > 0$ such that for $r \geq r_2$,
\[
G(r^{\rho^*_g(f)\varepsilon}) < F(r) < G(r^{\rho^*_g(f)\varepsilon}).
\]

(12)

From (11) and (12), we have for $r \geq r_3 = \max(r_1, r_2)$,
\[
\exp\{\{re^{L(r)}\}^{\rho^*_g(f)\varepsilon\varepsilon(1+\rho^*_g(f)\varepsilon)}\} < F(r)
\]
\[< \exp\{re^{L(r)}\}^{\rho^*_g(f)\varepsilon(1+\rho^*_g(f)\varepsilon)}\}.
\]

(13)

Since $\varepsilon > 0$ is arbitrary, from (13) we obtain that
\[
\rho^*_f = \lim_{r \to \infty} \frac{\log^2 F(r)}{\log[re^{L(r)}]} = \rho^*_g(f).
\]

Thus the theorem is established. ■
Theorem 3 If $f$ be of $L^\ast$-regular growth and of $L^\ast$-regular relative growth with respect to $g$ with rate $t$ for $t = 1, 2, 3, \ldots$ and $(t)\rho_g^{L^\ast}(f) = (t)\rho_f^{L^\ast}$ then $g$ is of $L^\ast$-regular growth of $L^\ast$-order one with rate $t$ in each case. Conversely if $g$ is of $L^\ast$-regular growth of $L^\ast$-order one with rate $t$ in each case then $(t)\rho_g^{L^\ast}(f) = (t)\rho_f^{L^\ast}$ for every entire $f$ with $L^\ast$-regular relative growth with rate $t$.

Proof. Let us first suppose that

$$(t)\rho_g^{L^\ast}(f) = (t)\rho_f^{L^\ast} = \rho > 0.$$  

Also let $0 < \varepsilon < 1$.  

Let us set $\varepsilon_1 = \frac{\rho \varepsilon}{2 + \varepsilon}$.  

So $\varepsilon_1 < \rho$.

Then there exists $r_0 > 0$ such that for $r \geq r_0$

$$F(r) < \exp\{r \exp[\ell L(r)]^{\rho + \varepsilon_1}\} \text{ and } F(r) > \exp\{r \exp[\ell L(r)]^{\rho - \varepsilon_1}\}.  \tag{14}$$

From (2) and (14) for $r \geq r_0$,

$$G(r^{\rho - \varepsilon_1}) < F(r) < \exp\{r \exp[\ell L(r)]^{\rho + \varepsilon_1}\}$$

and therefore for $r \geq r_0^\rho$ we obtain from above that

$$G(r) < \exp\{r \exp[\ell L(r)]^{\rho + \varepsilon_1}\} = \exp\{r \exp[\ell L(r)]^{1 + \frac{\varepsilon_1}{\rho - \varepsilon_1}}\}.$$ 

So, $G(r) < \exp\{r \exp[\ell L(r)]^{1+\varepsilon}\}$ for $r \geq r_0^\rho$.  \tag{15}

Similarly from (2) and (14) we obtain that

$$\exp\{r \exp[\ell L(r)]^{1-\varepsilon}\} < G(r) \text{ for } r \geq r_0^{2\rho}.  \tag{16}$$

So, from (15) and (16), for $r \geq r_0^{2\rho}$

$$\exp\{r \exp[\ell L(r)]^{1-\varepsilon}\} < G(r) < \exp\{r \exp[\ell L(r)]^{1+\varepsilon}\}.$$ 

So $g$ is of $L^\ast$-regular growth of order one with rate $t$. Conversely for $\varepsilon > 0$ there exists $r_1 > 0$ such that for $r \geq r_1$,

$$\exp\{r \exp[\ell L(r)]^{1-\varepsilon}\} < G(r) < \exp\{r \exp[\ell L(r)]^{1+\varepsilon}\}.  \tag{17}$$
Also from the definition of \( (t) \rho^L_g (f) \), there exists \( r_2 > 0 \) such that for \( r \geq r_2 \),

\[
G(r (t) \rho^L_g (f)^{-\varepsilon}) < F(r) < G(r (t) \rho^L_g (f)^{+\varepsilon}).
\]

From (17) and (18), we have for \( r \geq r_3 = \max(r_1, r_2) \),

\[
\exp\left\{r \exp[t] L(r)\right\}^{(t) \rho^L_g (f)^{-\varepsilon}(1+(t) \rho^L_g (f)^{-\varepsilon})} < F(r)
\]

\[
< \exp\left\{r \exp[t] L(r)\right\}^{(t) \rho^L_g (f)^{+\varepsilon}(1+(t) \rho^L_g (f)^{+\varepsilon})}.
\]

Since \( \varepsilon(> 0) \) is arbitrary, from (19) we obtain that

\[
(t) \rho^L_f = \lim_{r \to \infty} \frac{\log[2] F(r)}{\log[r \exp[t] L(r)]} = (t) \rho^L_g (f).
\]

This proves the theorem. \( \blacksquare \)

**Remark 1** For \( t = 1 \), Theorem 3 coincides with Theorem 2.

**References**


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