A New Hilbert-type Integral Inequality
with the Combination Kernel

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Abstract
By using the way of weight function, a new Hilbert-type integral inequality with a combination kernel and a best constant factor is given, which is an extension of a Hilbert-type integral inequality. As applications, the equivalent form and the reverse forms are considered.

Mathematics Subject Classification: 26D15

Keywords: Hilbert-type integral inequality; weight function; parameter

1 Introduction
If \( 0 < \int_{0}^{\infty} f^2(x)dx \leq \infty \) and \( 0 < \int_{0}^{\infty} g^2(x)dx < \infty \); then we have\([1]\):
\[
\int_{0}^{\infty} \int_{0}^{\infty} \frac{f(x)g(y)}{x+y}dxdy < \pi \left( \int_{0}^{\infty} f^2(x)dx \right) \left( \int_{0}^{\infty} g^2(x)dx \right)^{\frac{1}{2}};
\] (1)
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}} \, dx \, dy < 4 \left( \int_0^\infty f^2(x) \, dx \right)^{\frac{1}{2}},
\]
where the constant factor 4 are all the best possible. We call (1) Hilbert’s integral inequality. Both (1) and (2) are important in analysis and its applications\[1, 2\]. In recent years, by using the way of weight function, a number of extensions of (1) and (2) were given by Yang et al. \[3, 4\]. In 2006, Li et al. \[5\] gave the following inequality with a kernel relating (1) and (2):

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}} \, dx \, dy < c \left( \int_0^\infty f^2(x) \, dx \right)^{\frac{1}{2}},
\]
where the constant factor \( c = 2\sqrt{2 \arctan \frac{1}{\sqrt{2}}} \) is the best possible. In 2007, Xie \[6\] gave a best extension of (3).

In this paper, by introducing some parameters and using the way of weight function, we give a new Hilbert-type integral inequality with the combination kernel as

\[
1 + \max\{x^{\lambda}, y^{\lambda}\} + A(xy)^{\lambda/2} (\lambda > 0, A \geq 0),
\]
which is an extension of (2). As applications, the equivalent form and the reverse forms are obtained.

\section{Some Lemmas}

\textbf{Lemma 1.} \[\text{Lemma 1.}\] If \( \lambda > 0, A \geq 0, \) then

\[
k_\lambda(A) := \int_0^\infty \frac{u^{\lambda/2 - 1} \, du}{\max\{u^\lambda, 1\} + Au^{\lambda/2}} = \begin{cases} 
\frac{4}{\lambda^A} \ln(1 + A), & A > 0 \\
\frac{4}{\lambda}, & A = 0;
\end{cases} \tag{4}
\]

\[
\varpi_\lambda(y) := \int_0^\infty \frac{y^{\lambda/2} x^{\lambda/2 - 1} \, dx}{\max\{x^\lambda, y^\lambda\} + A(xy)^{\lambda/2}} = k_\lambda(A) (y \in (0, \infty)). \tag{5}
\]

\textbf{Proof.} Setting \( v = u^{\lambda/2} \), we find

\[
k_\lambda(A) = \frac{2}{\lambda} \left[ \int_0^1 \frac{dv}{1 + Av} + \int_1^\infty \frac{dv}{v^2 + Av} \right] = \frac{4}{\lambda} \int_0^1 \frac{dv}{1 + Av}.
\]

Hence (4) is valid. Setting \( u = x/y \), we obtain (5). The lemma is proved.

\textbf{Lemma 2.} \[\text{Lemma 2.}\] Assume that \( p > 0, |q| > 0, \lambda > 0, A \geq 0 \) and \( 0 < \varepsilon < \frac{\lambda}{2} \min\{p, |q|\} \). Then we have

\[
\int_0^1 \frac{u^{\lambda/2 + \frac{\varepsilon}{p} - 1} \, du}{1 + Au^{\lambda/2}} = \int_0^1 \frac{u^{\lambda/2 - 1} \, du}{1 + Au^{\lambda/2}} + o_1(\varepsilon \to 0^+); \tag{6}
\]
Hence Expressions (6), (7) and (8) are valid. The lemma is proved.

3 Main Results and Applications

Theorem 3. If \( \lambda, p > 0, \frac{1}{p} + \frac{1}{q} = 1, A \geq 0, \phi_r(x) = x^{r(1-\frac{1}{2})-1}(r = p, q), f, g \geq 0, 0 < ||f||_{p, \phi_p} = \{ \int_0^\infty \phi_p(x)f^p(x)dx \}^{\frac{1}{p}} < \infty \) and \( 0 < ||g||_{q, \phi_q} < \infty \), then

(a) for \( p > 1 \), we have the following equivalent inequalities:

\[
I_{\lambda} := \int_0^\infty y^{\frac{1}{2}-1}\left[ \int_0^{x^{\lambda}} \frac{f(x)dx}{\max\{x^\lambda, y^\lambda\} + A(xy)^{\frac{1}{2}}} \right]^pdy < k_{\lambda}(A)||f||_{p, \phi_p}; \quad (9)
\]

\[
J_{\lambda} := \int_0^\infty \int_0^{x^{\lambda}} \frac{f(x)g(y)dxdy}{\max\{x^\lambda, y^\lambda\} + A(xy)^{\lambda/2}} < k_{\lambda}(A)||f||_{p, \phi_p}||g||_{q, \phi_q}; \quad (10)
\]

(b) for \( 0 < p < 1 \), we have the reverse equivalent forms of (9) and (10).

Proof. By H"{o}lder's inequality and (5), for \( y \in (0, \infty) \), we obtain

\[
\left[ \int_0^{x^{\lambda}} \frac{f(x)dx}{\max\{x^\lambda, y^\lambda\} + A(xy)^{\lambda/2}} \right]^p
\]
Setting reverse equivalent forms of (9) and (10). The theorem is proved.

Therefore we have (9), which is equivalent to (10). In view of (9), we have (10).

\[ \int_0^\infty \frac{1}{\max\{x^\lambda, y^\lambda\} + A(x(y)^{1/2}/f(x))}\left[\frac{y^{(1-\frac{1}{2})/p}}{x^{(1-\frac{1}{2})/q}}\right]dx \right]^p \]

\[ \leq \int_0^\infty \frac{1}{\max\{x^\lambda, y^\lambda\} + A(x(y)^{1/2})}\left[\int_0^\infty \frac{y^{(1-\frac{1}{2})/(q-1)}x^{1/2-1}}{\max\{x^\lambda, y^\lambda\} + A(x(y)^{1/2})}\right]dx \right]^{-p-1} = k_{\lambda}^{p-1}(A)\int_0^\infty \frac{y^{1-\frac{1}{2}}}{\max\{x^\lambda, y^\lambda\} + A(x(y)^{1/2})}f^p(x)dx. \quad (11) \]

By (5), in view of Fubini’s Theorem, it follows

\[ I_{\lambda} \leq k_{\lambda}^{p-1}(A)\int_0^\infty \int_0^\infty \frac{x^{(1-\frac{1}{2})(p-1)}y^{1/2-1}f^p(x)}{\max\{x^\lambda, y^\lambda\} + A(x(y)^{1/2})}dxdy \]

\[ = k_{\lambda}^{p-1}(A)\int_0^\infty \varphi(x)\phi(y)f^p(x)dx = k_{\lambda}^{p}(A)||f||^{p}_{p,\phi_p}. \quad (12) \]

If (11) takes equality, then there exists \( A \) and \( B \), such that they are not all zero and \( A(x^{(1-\frac{1}{2})(p-1)}y^{1/2-1}f^p(x) = By^{(1-\frac{1}{2})(q-1)}x^{1/2-1} \) a.e. in \((0, \infty)\). It means \( A(x^{p(1-\frac{1}{2})}f^p(x) = By^{q(1-\frac{1}{2})} \) a.e. in \((0, \infty)\). We affirm that \( A \neq 0 \), otherwise \( B = A = 0 \). Hence it follows \( x^{p(1-\frac{1}{2})}f^p(x) = By^{q(1-\frac{1}{2})/Ax} \) a.e. in \((0, \infty)\), which contradicts \( 0 < ||f||_{p,\phi_p} < \infty \). Then (11) and (12) keep the strict inequalities, and we have (9). By Hölder’s inequality\(^7\), we find

\[ J_{\lambda} = \int_0^\infty \left[ \int_0^\infty \frac{y^{1/2}(x^{1/2})f^p(x)dx}{\max\{x^\lambda, y^\lambda\} + A(x(y)^{1/2})}\right]^{1/2}dy \leq I_{\lambda}^{1/2} ||g||^{1/2}_{q,\phi_q}. \quad (13) \]

In view of (9), we have (10).

On the other-hand, suppose (10) is valid. We find \( I_{\lambda} > 0 \), since \( ||f||_{p,\phi_p} > 0 \). If \( I_{\lambda} = \infty \), then (9) is not valid, since \( ||f||_{p,\phi_p} < \infty \). Hence we have \( 0 < I_{\lambda} < \infty \). Setting \( g(y) := y^{\frac{1}{2}}(\int_0^\infty \frac{f(x)dx}{\max\{x^\lambda, y^\lambda\} + A(x(y)^{1/2})})^{p-1}, y \in (0, \infty) \), by (10), we find

\[ 0 < \int_0^\infty y^{\frac{q(1-\frac{1}{2})}{p}}g^q(y)dy = I_{\lambda} = J_{\lambda} \]

\[ < k_{\lambda}(A)||f||_{p,\phi_p} \left( \int_0^\infty y^{\frac{q(1-\frac{1}{2})}{p}}g^q(y)dy \right)^{\frac{1}{p}} < \infty; \quad (14) \]

\[ I_{\lambda} = \int_0^\infty y^{\frac{q(1-\frac{1}{2})}{p}}g^q(y)dy < k_{\lambda}^{p}(A)||f||^{p}_{p,\phi_p}. \quad (15) \]

Therefore we have (9), which is equivalent to (10).

(b) By the reverse Hölder’s inequality and the same way, we can obtain the reverse equivalent forms of (9) and (10). The theorem is proved.
Theorem 4. As the assumption of Theorem 1, all the constant factors in (9), (10) and the reverse forms are the best possible.

Proof. For $0 < \varepsilon < \frac{1}{2} \min\{p, |q|\}$, we set $f_\varepsilon, g_\varepsilon$ as: $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0, 1)$; $f_\varepsilon(x) = x^{\frac{1}{2} + \frac{\varepsilon}{2}}, g_\varepsilon(x) = x^{\frac{1}{2} - \frac{\varepsilon}{2}},$ for $x \in [1, \infty)$.

(a) For $p > 1$, if there exists a constant $0 < k \leq k_A$, such that (10) is still valid as we replace $k_\lambda(A)$ by $k$, then in particular, we find

$$k = \varepsilon k ||f_\varepsilon||_{p, \phi_p} ||g_\varepsilon||_{q, \phi_q} > \varepsilon \int_0^\infty \int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)dxdy}{\max\{x^\lambda, y^\lambda\} + A(xy)^\frac{1}{2}}$$

$$= \varepsilon \int_1^\infty x^{\frac{1}{2} - \frac{\varepsilon}{2} - 1}[\int_1^\infty \frac{y^{\frac{1}{2} + \frac{\varepsilon}{2} - 1}dy}{\max\{x^\lambda, y^\lambda\} + A(xy)^\frac{1}{2}}]dx.$$ 

Setting $u = x/y$ in the above integral, by Fubini’s Theorem, we have

$$k > \varepsilon \int_1^\infty x^{-\varepsilon - 1}[\int_0^x \frac{u^{\frac{1}{2} + \frac{\varepsilon}{2} - 1}}{\max\{x^\lambda, 1\} + Au^\lambda/2}du]dx$$

$$= \int_0^1 \frac{u^{\frac{1}{2} + \frac{\varepsilon}{2} - 1}}{1 + Au^\lambda/2}du + \varepsilon \int_1^\infty x^{-\varepsilon - 1}[\int_0^x \frac{u^{\frac{1}{2} + \frac{\varepsilon}{2} - 1}}{u^\lambda + Au^\lambda/2}du]dx$$

$$= \int_0^1 \frac{u^{\frac{1}{2} + \frac{\varepsilon}{2} - 1}}{1 + Au^\lambda/2}du + \varepsilon \int_1^\infty \frac{x^{\frac{1}{2} - \frac{\varepsilon}{2} - 1}dx}{u^\lambda + Au^\lambda/2}du$$

$$= \int_0^1 \frac{u^{\frac{1}{2} + \frac{\varepsilon}{2} - 1}}{1 + Au^\lambda/2}du + \int_1^\infty \frac{u^{\frac{1}{2} - \frac{\varepsilon}{2} - 1}du}{u^\lambda + Au^\lambda/2}.$$ 

For $\varepsilon \to 0^+$, in view of (6) and (7), we find $k \geq k_A$. Hence $k = k_A$ is the best constant factor of (10). If the constant factor in (9) is not the best possible, then by (13), we may get a contradiction that the constant factor in (10) is not the best possible.

(b) For $0 < p < 1$, if there exists $K \geq k_A$, such that the reverse form of (10) is valid as we replace $k_\lambda(A)$ by $K$, then we have

$$K = \varepsilon K ||f_\varepsilon||_{p, \phi_p} ||g_\varepsilon||_{q, \phi_q} < \varepsilon \int_0^\infty \int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)dxdy}{\max\{x^\lambda, y^\lambda\} + A(xy)^\frac{1}{2}}$$

$$\leq \varepsilon \int_1^\infty y^{\frac{1}{2} - \frac{\varepsilon}{2} - 1}[\int_0^\infty \frac{x^{\frac{1}{2} - \frac{\varepsilon}{2} - 1}dx}{\max\{x^\lambda, y^\lambda\} + A(xy)^\frac{1}{2}}]dy$$

$$= \int_0^\infty \frac{u^{\frac{1}{2} - \frac{\varepsilon}{2} - 1}du}{\max\{u^\lambda, 1\} + Au^\lambda/2} \leq \int_0^1 \frac{u^{\frac{1}{2} - \frac{\varepsilon}{2} - 1}du}{1 + Au^\lambda/2} + \int_1^\infty \frac{u^{\frac{1}{2} - \frac{\varepsilon}{2} - 1}du}{u^\lambda + Au^\lambda/2}.$$
For $\varepsilon \to 0^+$, in view of (8), we obtain $K \leq k_\lambda(A)$. Hence $K = k_\lambda(A)$ is the best constant factor of the reverse form of (10). If the constant factor in the reverse form of (9) is not the best possible, then by the reverse form of (13), we may get a contradiction that the constant factor in the reverse form of (10) is not the best possible. The theorem is proved.

**Remark 1 Remarks.** (a) For $p = q = 2, \lambda = 1, A = 0$ in (10), we have (2). Hence inequality (10) is an extension of (2). (b) for $A = 1, \lambda > 0$ in (10), we obtain a new inequality with the best constant factor $(4 \ln 2)/\lambda$ as follows:

$$
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\} + (xy)^{\frac{\lambda}{2}}}dxdy < \frac{4\ln 2}{\lambda} ||f||_{p,\phi_p} ||g||_{q,\phi_q}.
$$

(16)

**References**


Received: December, 2010