A Common Unique Random Fixed Point Theorem for Expansive Type Mappings in Hilbert Space

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Abstract

The object of this paper is to obtain a common unique fixed point theorem for two continuous, surjective random operators defined on a non empty closed subset of a separable Hilbert space. The corresponding result for non random case is also obtained.

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1 Introduction

The study of random fixed points has been an active area of contemporary research in mathematics. Some of the recent works in this field are noted in [2, 4, 5, 1]. In this paper, we construct a sequence of measurable functions and consider its convergence to the common unique random fixed point of two continuous, surjective random operators defined on a non-empty closed subset of a separable Hilbert space. For the purpose of obtaining the random fixed point of two continuous, surjective random operators, we have used a rational inequality.

Throughout this paper, \((\Omega, \Sigma)\) denotes a measurable space consisting of a set \(\Omega\) and sigma algebra \(\Sigma\) of subsets of \(\Omega\), \(H\) stands for a separable Hilbert space and \(C\) is a nonempty closed subset of \(H\).
2 Preliminaries

Definition 2.1. A function \( f : \Omega \to C \) is said to be measurable if 
\( f^{-1}(B \cap C) \in \Sigma \) for every Borel subset \( B \) of \( H \).

Definition 2.2. A function \( F : \Omega \times C \to C \) is said to be a random operator if \( F(., x) : \Omega \to C \) is measurable for every \( x \in C \).

Definition 2.3. A measurable function \( g : \Omega \to C \) is said to be a random fixed point of the random operator \( F : \Omega \times C \to C \) if \( F(t, g(t)) = g(t) \) for all \( t \in \Omega \).

Definition 2.4. A random operator \( F : \Omega \times C \to C \) is said to be continuous if for fixed \( t \in \Omega \), \( F(t, .) : C \to C \) is continuous.

Condition (A). Two mappings \( S, T : C \to C \), where \( C \) is a non-empty subset of a Hilbert space \( H \), is said to satisfy condition (A) if

\[
\| Sx - Ty \|^2 \geq a_1 \| x - Sx \|^2 \| y - Ty \|^2 + a_2 \frac{\| x - Ty \|^2 \| y - Sx \|^2}{\| x - g \|^2} + a_3 \| x - Sx \|^2 + a_4 \| y - Ty \|^2 + a_5 \| x - y \|^2
\]

where \( a_1 + a_3 + a_4 + a_5 > 1 \), \( a_2 + a_5 > 1 \) and \( a_1, a_2, a_3, a_4, a_5 > 0 \) \hfill (2.1)

We construct a sequence of functions \( \{g_n\} \) as

\[
g_0 : \Omega \to C
\]

is arbitrary measurable function. For \( t \in \Omega \) and \( n = 0, 1, 2, \ldots \)

\[
g_{2n}(t) = S(t, g_{2n+1}(t)), \quad g_{2n+1}(t) = T(t, g_{2n+2}(t)) \hfill (2.4)
\]

3 Main Results

Theorem 3.1. Let \( C \) be a non-empty closed subset of a separable Hilbert space \( H \). Let \( S \) and \( T \) be two continuous, surjective random operators defined on \( C \) such that for \( t \in \Omega \), \( S(t, .), T(t, .) : C \to C \) satisfy condition (A). Then the sequence \( \{g_n\} \) obtained in (2.3) and (2.4) converges to the unique common random fixed point of \( S \) and \( T \).

Proof: For fixed \( t \in \Omega \), \( n = 1, 2, 3, \ldots \)

\[
\| g_{2n-1}(t) - g_{2n}(t) \|^2 = \| T(t, g_{2n}(t)) - S(t, g_{2n+1}(t)) \|^2 = \| S(t, g_{2n+1}(t)) - T(t, g_{2n}(t)) \|^2
\]
\[ \geq a_1 \frac{\|g_{2n-1}(t) - S(t, g_{2n-1}(t))\|^2 \|g_{2n}(t) - T(t, g_{2n}(t))\|^2}{\|g_{2n-1}(t) - g_{2n}(t)\|^2} + a_2 \frac{\|g_{2n+1}(t) - T(t, g_{2n+1}(t))\|^2 \|g_{2n}(t) - S(t, g_{2n+1}(t))\|^2}{\|g_{2n+1}(t) - g_{2n}(t)\|^2} \\
+ a_3 \left( \|g_{2n+1}(t) - S(t, g_{2n+1}(t))\|^2 \right) + a_4 \left( \|g_{2n}(t) - T(t, g_{2n}(t))\|^2 \right) + a_5 \left( \|g_{2n+1}(t) - g_{2n}(t)\|^2 \right) \\
= a_1 \frac{\|g_{2n+1}(t) - g_{2n}(t)\|^2 \|g_{2n}(t) - g_{2n-1}(t)\|^2}{\|g_{2n+1}(t) - g_{2n}(t)\|^2} + a_2 \frac{\|g_{2n+1}(t) - g_{2n-1}(t)\|^2 \|g_{2n}(t) - g_{2n}(t)\|^2}{\|g_{2n+1}(t) - g_{2n}(t)\|^2} \\
+ a_3 \left( \|g_{2n+1}(t) - g_{2n}(t)\|^2 \right) + a_4 \left( \|g_{2n}(t) - g_{2n-1}(t)\|^2 \right) + a_5 \left( \|g_{2n+1}(t) - g_{2n}(t)\|^2 \right) \\
= (a_1 + a_4) \left( \|g_{2n}(t) - g_{2n-1}(t)\|^2 \right) + (a_3 + a_5) \left( \|g_{2n+1}(t) - g_{2n}(t)\|^2 \right) \\
\Rightarrow \|g_{2n}(t) - g_{2n+1}(t)\|^2 \leq \frac{1 - (a_1 + a_4)}{a_3 + a_5} \left( \|g_{2n-1}(t) - g_{2n}(t)\|^2 \right) \\
\Rightarrow \|g_{2n}(t) - g_{2n+1}(t)\|^2 \leq \left( \frac{1 - (a_1 + a_4)}{a_3 + a_5} \right)^2 \left( \|g_{2n-1}(t) - g_{2n}(t)\|^2 \right) \\
\Rightarrow \|g_{2n}(t) - g_{2n+1}(t)\| \leq k_1 \left\| g_{2n-1}(t) - g_{2n}(t) \right\| \\
\text{where } k_1 = \left( \frac{1 - (a_1 + a_4)}{a_3 + a_5} \right)^\frac{1}{2} < 1 \quad \text{(as } a_1 + a_3 + a_4 + a_5 > 1) \\
\]

For fixed \( t \in \Omega, n = 1, 2, 3, \ldots \)

\[ \|g_{2n-2}(t) - g_{2n-1}(t)\|^2 = \|S(t, g_{2n-1}(t)) - T(t, g_{2n}(t))\|^2 \]

\[ \geq a_1 \frac{\|g_{2n-1}(t) - S(t, g_{2n-1}(t))\|^2 \|g_{2n}(t) - T(t, g_{2n}(t))\|^2}{\|g_{2n-1}(t) - g_{2n}(t)\|^2} + a_2 \frac{\|g_{2n-1}(t) - T(t, g_{2n}(t))\|^2 \|g_{2n}(t) - S(t, g_{2n-1}(t))\|^2}{\|g_{2n-1}(t) - g_{2n}(t)\|^2} \\
+ a_3 \left( \|g_{2n-1}(t) - S(t, g_{2n-1}(t))\|^2 \right) + a_4 \left( \|g_{2n}(t) - T(t, g_{2n}(t))\|^2 \right) + a_5 \left( \|g_{2n-1}(t) - g_{2n}(t)\|^2 \right) \\
= a_1 \frac{\|g_{2n-1}(t) - g_{2n-2}(t)\|^2 \|g_{2n}(t) - g_{2n-1}(t)\|^2}{\|g_{2n-1}(t) - g_{2n}(t)\|^2} + a_2 \frac{\|g_{2n-1}(t) - g_{2n-2}(t)\|^2 \|g_{2n}(t) - g_{2n}(t)\|^2}{\|g_{2n-1}(t) - g_{2n}(t)\|^2} \\
+ a_3 \left( \|g_{2n-1}(t) - g_{2n-2}(t)\|^2 \right) + a_4 \left( \|g_{2n}(t) - g_{2n-1}(t)\|^2 \right) + a_5 \left( \|g_{2n-1}(t) - g_{2n}(t)\|^2 \right) \\
= (a_1 + a_3) \left( \|g_{2n-1}(t) - g_{2n-2}(t)\|^2 \right) + (a_4 + a_5) \left( \|g_{2n-1}(t) - g_{2n}(t)\|^2 \right) \\
\Rightarrow \|g_{2n-1}(t) - g_{2n}(t)\|^2 \leq \frac{1 - (a_1 + a_3)}{a_4 + a_5} \left( \|g_{2n-2}(t) - g_{2n-1}(t)\|^2 \right) \\
\Rightarrow \|g_{2n-1}(t) - g_{2n}(t)\|^2 \leq \left( \frac{1 - (a_1 + a_3)}{a_4 + a_5} \right)^2 \left( \|g_{2n-2}(t) - g_{2n-1}(t)\|^2 \right) \\
\Rightarrow \|g_{2n-1}(t) - g_{2n}(t)\| \leq k_2 \left\| g_{2n-2}(t) - g_{2n-1}(t) \right\| \\
\text{where } k_2 = \left( \frac{1 - (a_1 + a_3)}{a_4 + a_5} \right)^\frac{1}{2} < 1 \quad \text{(as } a_1 + a_3 + a_4 + a_5 > 1) \\
\]

The inequalities (3.1) and (3.2) are jointly imply that for all \( t \in \Omega, n = 1, 2, 3, \ldots \)
\[ \| g_n(t) - g_{n+1}(t) \| \leq k \| g_{n-1}(t) - g_n(t) \| \]

where \( k = \max \{ k_1, k_2 \} < 1 \)

\[ \Rightarrow \| g_n(t) - g_{n+1}(t) \| \leq k^n \| g_0(t) - g_1(t) \| \quad \text{for} \ t \in \Omega \] (3.3)

Now, we shall prove that for \( t \in \Omega \), \( \{g_n(t)\} \) is a Cauchy sequence. For this for every positive integer \( p \), we have

\[ \| g_n(t) - g_{n+p}(t) \| = \| g_n(t) - g_{n+1}(t) + g_{n+1}(t) - \ldots + g_{n+p-1}(t) - g_{n+p}(t) \| \]

\[ \leq \| g_{n+1}(t) - g_{n+2}(t) \| + \ldots + \| g_{n+p-1}(t) - g_{n+p}(t) \| \]

\[ \leq \left[ k^n + k^{n+1} + k^{n+2} + \ldots + k^{n+p-1} \right] \| g_0(t) - g_1(t) \| \quad \text{(by (3.3))} \]

\[ = k^n \left[ 1 + k + k^2 + \ldots + k^{p-1} \right] \| g_0(t) - g_1(t) \| \]

\[ < \frac{k^n}{1-k} \| g_0(t) - g_1(t) \| \quad \text{for} \ t \in \Omega \]

As \( n \to \infty \), \( \| g_n(t) - g_{n+p}(t) \| \to 0 \), it follows that for \( t \in \Omega \), \( \{g_n(t)\} \) is a Cauchy sequence and hence is convergent in Hilbert space \( H \).

For \( t \in \Omega \), let

\[ \{g_n(t)\} \to g(t) \quad \text{as} \ n \to \infty \] (3.4)

Since \( C \) is closed, \( g \) is a function from \( C \) to \( C \).

Since \( S \) and \( T \) are surjective maps. So there exist two functions \( g' : \Omega \to C \) and \( g'' : \Omega \to C \) such that

\[ g(t) = S(t,g'(t)) \text{ and } g(t) = T(t,g''(t)) \] (3.5)

For \( t \in \Omega \),

\[ \| g_{2n}(t) - g(t) \|^2 = \| S(t,g_{2n+1}(t)) - T(t,g''(t)) \|^2 \]

\[ \geq a_1 \| g_{2n+1}(t)-S(t,g_{2n+1}(t)) \|^2 g''(t)-T(t,g''(t)) \|^2 + a_2 \| g_{2n+1}(t)-T(t,g''(t)) \|^2 g''(t)-S(t,g_{2n+1}(t)) \|^2 \]

\[ + a_3 \| g_{2n+1}(t)-S(t,g_{2n+1}(t)) \|^2 + a_4 \| g''(t)-T(t,g''(t)) \|^2 + a_5 \| g_{2n+1}(t)-g''(t) \|^2 \]

\[ = a_1 \| g_{2n+1}(t)-g_{2n}(t) \|^2 g''(t)-T(t,g''(t)) \|^2 + a_2 \| g_{2n+1}(t)-T(t,g''(t)) \|^2 g''(t)-g_{2n}(t) \|^2 \]

\[ + a_3 \| g_{2n+1}(t)-g_{2n}(t) \|^2 + a_4 \| g''(t)-T(t,g''(t)) \|^2 + a_5 \| g_{2n+1}(t)-g''(t) \|^2 \]
Making $n \to \infty$ in the above inequality we have by virtue of (3.4), for all $t \in \Omega$,

$$0 \geq (a_4 + a_5) \| g(t) - g''(t) \|^2$$

$$\Rightarrow \| g(t) - g''(t) \|^2 = 0 \quad \text{[as } (a_4 + a_5) > 0\text{]}$$

$$\Rightarrow g(t) = g''(t) \text{ for } t \in \Omega \quad (3.6)$$

In an exactly similar way by using $(a_3 + a_5) > 0$ we can prove that

$$g(t) = g'(t) \text{ for } t \in \Omega \quad (3.7)$$

Thus by (3.5), (3.6) and (3.7) we have, for $t \in \Omega$,

$$S(t, g(t)) = g(t) \quad (3.8)$$

and

$$T(t, g(t)) = g(t) \quad (3.9)$$

Again, if $A : \Omega \times C \to C$ is a continuous random operator on a nonempty subset $C$ of a separable Hilbert space $H$, then for any measurable function $f : \Omega \to C$, the function $h(t) = A(t, f(t))$ is also measurable [3].

It follows from the construction of $\{g_n\}$ ((2.3) and (2.4)) and the above consideration that $\{g_n\}$ is a sequence of measurable functions. From (3.4) it follows that $g$ is also a measurable function. This fact along with ((3.8) and (3.9)) shows that $g : \Omega \to C$ is a common random fixed point of $S$ and $T$.

Next we prove the uniqueness. Let $h : \Omega \to C$ be another random fixed point common to $S$ and $T$, that is, for $t \in \Omega$,

$$S(t, h(t)) = h(t) \text{ and } T(t, h(t)) = h(t) \quad (3.10)$$

Then for $t \in \Omega$,

$$\| g(t) - h(t) \|^2 = \| S(t, g(t)) - T(t, h(t)) \|^2$$

$$\geq a_1 \frac{\| g(t) - S(t, g(t)) \|^2 \| h(t) - T(t, h(t)) \|^2}{\| g(t) - h(t) \|^2} + a_2 \frac{\| g(t) - T(t, h(t)) \|^2 \| h(t) - S(t, g(t)) \|^2}{\| g(t) - h(t) \|^2}$$

$$+ a_3 \| g(t) - S(t, g(t)) \|^2 + a_4 \| h(t) - T(t, h(t)) \|^2 + a_5 \| g(t) - h(t) \|^2$$
\[ \Rightarrow \| g(t) - h(t) \|^2 \geq (a_2 + a_5) \| g(t) - h(t) \|^2 \text{ (by (3.10))} \]

\[ \Rightarrow \| g(t) - h(t) \|^2 = 0 \quad \text{[as } (a_2 + a_5) > 1] \]

\[ \Rightarrow g(t) = h(t) \text{ for all } t \in \Omega \]

This completes the proof of the theorem 3.1.

**Corollary 3.2.** Let \( S, T : C \rightarrow C \) be two continuous, surjective self maps of a nonempty closed subset \( C \) of a Hilbert Space \( H \), be such that inequality (2.1) is satisfied along with (2.2). Then the sequence obtained by starting with an arbitrary element \( x_0 \in C \),

\[
\begin{align*}
x_{2n} &= Sx_{2n+1}, \quad n = 0, 1, 2, \ldots \quad (3.11) \\
x_{2n+1} &= Tx_{2n+2}, \quad n = 0, 1, 2, \ldots \quad (3.12)
\end{align*}
\]

converges to a unique common fixed point of \( S \) and \( T \).

The proof of the corollary is immediate by assuming \( \Omega \) to be a singleton set.

**Remark 3.3.** It is necessary to assume \( H \) to be separable in the corollary (3.2).

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**References**


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