The Space of Maximal Ideals in an Almost Distributive Lattice

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Abstract

This paper deals with the space \( \mathcal{P} \) of prime ideals, the space \( \mathcal{M} \) of maximal ideals and the space \( \Sigma \) of all prime ideals containing all dual dense elements of an ADL \( R \) with 0 and maximal elements. The aim of this paper is to characterize the regularity of the space \( \mathcal{P} \) and the normality of the spaces \( \mathcal{M} \) and \( \Sigma \).

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1 Introduction

It is well known that the space of prime ideals of a bounded distributive lattice \( L \) is a compact, \( T_0 \)-space with respect to the hull kernel topology and that this space satisfies several interesting properties. Many lattice theoretic properties of \( L \) are characterized in terms of certain topological properties of this space. In particular it is well known that \( L \) is a Boolean algebra iff the space of prime ideals of \( L \) is a Hausdorff space. Almost distributive lattice arise as a natural generalization of a distributive lattices and hence it is natural to consider the properties of the space of prime
ideals in an almost distributive lattice. It is interesting to note that the results which are valid for distributive lattices in verbatim, for ADLs, even though the techniques of the proofs in the case of ADLs are slightly different, for the reason that the operations $\land$ and $\lor$ are not commutative.

For an ADL $R$ with 0 and maximal elements, let $\wp$ denote the set of all prime ideals in $R$. The prime spectrum of $R$ is the set $\wp$ equipped with the hull kernel topology. Some necessary and sufficient conditions for the space $\wp$ to be normal in a dually semi-complemented ADL $R$ are proved.

Let $\mathfrak{m}$ denote the set of all maximal ideals in $R$. Then obviously, $\mathfrak{m} \subseteq \wp$. The maximal spectrum of $R$ is the set $\mathfrak{m}$ equipped with the hull kernel topology. Some equivalent conditions for the space $\mathfrak{m}$ to be normal are furnished. As a special case, the properties of $\mathfrak{m}$ in a dually semi-complemented ADL are proved. Let $\Sigma = \{ P \in \wp \mid \cap \mathfrak{m} \subseteq P \}$ i.e. the set $\Sigma$ is the set of all prime ideals consisting of all dual dense elements in $R$. A detailed study of the special subspace $\Sigma$ of $\wp$ is carried out in this paper. Several algebraic properties of $R$ are characterized in terms of certain topological properties of the spaces $\Sigma$ and $\mathfrak{m}$. In particular, it is proved that The maximal spectrum $\mathfrak{m}$ of $R$ is a retract of $\Sigma$ iff each $P \in \Sigma$ is contained in a unique maximal ideal $M$ in $\mathfrak{m}$. Also it is observed that the space $\mathfrak{m}$ is normal iff the space $\Sigma$ is normal. Finally, we have given a necessary and sufficient condition for the space $\wp$ to be regular space.

2 Preliminary Notes

We first recall some basic concepts and important results for the sake of completeness. Recall from [1] an almost distributive lattice (ADL) with 0 is an algebra $< R, \lor, \land, 0 >$ of the type $(2, 2, 0)$ satisfying

1. $a \lor 0 = a$,
2. $0 \land a = 0$,
3. $(a \lor b) \land c = (a \land c) \lor (b \land c)$,
4. $a \land (b \lor c) = (a \land b) \lor (a \land c)$,
5. $a \lor (b \land c) = (a \lor b) \land (a \lor c)$,
6. $(a \lor b) \land b = b$, for all $a, b, c \in R$

Let $X$ be any non-empty set. Fix $x_0 \in X$. For any $x, y \in X$, define binary operations $\lor, \land$ on $X$ by

$$x \lor y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \quad x \land y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0 \end{cases}$$
Then \((X, \lor, \land, x_0)\) is an ADL which is called a discrete ADL with zero \(x_0\).
If \(< R, \lor, \land, 0 >\) is an ADL, for any \(a, b \in R\), define \(a \leq b\) if and only if \(a = a \land b\) or, equivalently, \(a \lor b = b\), then \(\leq\) is a partial ordering on \(R\).
A non-empty subset \(I\) of \(R\) is said to be an ideal (filter) of \(R\), if it satisfies the following;

(i) \(a, b \in I \Rightarrow a \lor b \in I\) (\(a \land b \in I\))
(ii) \(a \in I, x \in R \Rightarrow a \land x \in I\) (\(a \lor x \in I\)).
If \(I\) is an ideal of \(R\) and \(a, b \in R\), then \(a \land b \in I \iff b \land a \in I\).
A proper ideal (filter) \(P\) of \(R\) is said to be maximal if, there is no proper ideal (filter) \(Q\) of \(R\) such that \(P \subset Q\).
If \(P\) is a proper ideal of \(R\), then we say that \(P\) is prime ideal if for any \(x, y \in R\), \(x \land y \in P \Rightarrow x \in P\) or \(y \in P\). Analogously, we can define the concept of a prime filter.
An element \(m \in R\) is called maximal if it is a maximal element in the partially ordered set \((R, \leq)\). That is for any \(a \in R\), \(m \leq a \Rightarrow m = a\).
An element \(x \in R\) is said to be a dual dense elements if \((x)^+ = A\), where \(A\) is the set of all maximal elements in \(R\) and \((x)^+ = \{y \in R \mid x \lor y = m\},\) for some \(m \in A\). If \(E\) denote the set of all dual dense elements in \(R\), then \(E\) is an ideal in \(R\) and \(E = \bigcap\{M \mid M \in \mathcal{M}\}\). [3]
Any two filters are weakly co-maximal, if \(F_1 \lor F_2\) contains a dual-dense element. [3]
An ADL \(R\) with maximal elements is called a dually semi-complemented if, for each non-zero element \(x \in R\), there exists a non-maximal element \(y\) of \(R\) such that \(x \lor y\) is maximal. [3]

Now we review some topological concepts.
Let \(X\) be a topological space. \(X\) is called \(T_0\) if distinct points points of \(X\) have distinct closures. A point \(p\) of \(X\) is called a \(T_1\) point if the closure of \(p\) contains no point other than \(p\). \(X\) is called \(T_1\) if every point of \(X\) is \(T_1\). \(X\) is called \(T_2\) if any two distinct points of \(X\) have disjoint neighbourhoods. A subset \(A\) of \(X\) is called compact, if every open cover of \(A\) has a finite subcover. A subset \(A\) of \(X\) is said to be dense if \(\text{Cl.} A = X\). \(X\) is called normal if given any two disjoint closed subsets \(C_1, C_2\) of \(X\) we can find subsets \(C_3, C_4\) of \(X\) containing \(C_1, C_2\) respectively such that \(C_1 \cap C_4 = \phi = C_2 \cap C_3\) and \(C_3 \cup C_4 = X\). A topological space \(X\) is a regular space if, given any nonempty closed set \(F\) and any point \(x\) that does not belong to \(F\), there exists a neighbourhood \(U\) of \(x\) and a neighbourhood \(V\) of \(F\) that are disjoint. Let \(E\) be a subspace of the space \(X\). A map \(\gamma : X \to E\) is called a retraction of \(X\) onto \(E\) if \(\gamma |_E\) is the identity map on \(E\). A subspace \(E\) of \(X\) is called a retract of \(X\) if there is a retraction of \(X\) onto \(E\).

**Result 2.1.** [4] For any \(a, b, c \in R\), we have the following:

1. \(a \lor b = a \iff a \land b = b\)
2. \( a \lor b = b \iff a \land b = a \)

3. \( \land \) is associative in \( R \)

4. \( a \land b \land c = b \land a \land c \)

5. \( (a \lor b) \land c = (b \lor a) \land c \)

6. \( a \land b = 0 \iff b \land a = 0 \)

7. \( a \land (b \lor a) = a \land (a \lor b) = a \)

8. \( a \leq a \lor b \) and \( a \land b \leq b \)

9. If \( a \leq c, b \leq c \) then \( a \land b = b \land a \) and \( a \lor b = b \lor a \)

10. \( a \lor b = (a \lor b) \lor a \)

11. \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \).

**Result 2.2.** [2] The set \( I(R) \) of all ideals of \( R \) is a complete distributive lattice with the least elements \( \{0\} \) and the greatest element \( R \) under set inclusion in which for any \( I, J \in I(R), I \cap J \) is the infimum of \( I \) and \( J \) and the supremum is given by \( I \lor J = \{i \lor j \mid i \in I, j \in J\} \).

**Result 2.3.** [2] For any subset \( S \) of \( R \) is the smallest ideal containing \( S \) is given by \( \langle S \rangle = \{(\lor_{i=1}^{n} s_i) \land x \mid s_i \in S, x \in R \) and \( n \in N\} \) and if \( a \in R \), then \( \langle a \rangle = \{a \land x \mid x \in R\} \) is ideal generated by \( a \). Similarly, for any \( a \in R \), \( \langle a \rangle = \{x \lor a \mid x \in R\} \) is the filter generated by \( a \).

**Result 2.4.** [2] A subset \( P \) of \( R \) is a prime ideal of \( R \) if and only if \( R \setminus P \) is a prime filter.

**Result 2.5.** [2] Every maximal ideal of \( R \) is a prime ideal.

**Result 2.6.** [2] Let \( I \) be an ideal and \( F \) is a filter of an ADL \( R \) with \( 0 \) such that \( I \cap F = \phi \). Then there exists a prime ideal \( P \) such that \( I \subseteq P \) and \( P \cap F = \phi \).

**Result 2.7.** [2] Let \( P \) be a prime ideal of an ADL \( R \) with \( 0 \). Then \( P \) is a minimal prime ideal iff for each \( x \in P \) there exists \( y \notin P \) such that \( x \land y = 0 \).

**Result 2.8.** [4] Let \( R \) be an ADL and \( m \in R \). Then the following are equivalent

1. \( m \) is the maximal with respect to \( \leq \).
2. \( m \lor a = m \), for all \( a \in R \).

3. \( m \land a = a \), for all \( a \in R \).

Result 2.9. [3] Let \( R \) be an ADL with maximal elements. Then \( R \) is a dually semi-complemented if and only if the intersection of all maximal ideal is \( \{0\} \).

Result 2.10. [3] Let \( R \) be an ADL with maximal elements then \( U(a) = \emptyset \) if and only if \( a \in E \).

3 Topological spaces

Throughout this paper we consider an ADL \( R \) with 0 and maximal elements. Let \( \varnothing \) denote the set of all prime ideals in \( R \). The prime spectrum of \( R \) is the set \( \varnothing \) equipped with the hull kernel topology. The set \( \mathcal{B} = \{ V(a) \mid a \in R \} \) where \( V(a) = \{ P \in \varnothing \mid a \notin P \} \) forms a base for the open sets for the space \( \varnothing \). The sets \( V(a)(a \in R) \) are the only compact open sets in \( \varnothing \). As for any maximal element \( m \) in \( R \), \( V(m) = \varnothing \), the space \( \varnothing \) is compact. For any \( \mathcal{F} \subseteq \varnothing \), the closure of \( \mathcal{F} \) in \( \varnothing \) is given by \( \text{Cl}_\varnothing \mathcal{F} = \{ P \in \varnothing \mid \bigcap \mathcal{F} \subseteq P \} \). Thus \( \text{Cl}_\varnothing \mathcal{F} = h(I) \) (hull of \( I \)) where \( I \) is an ideal in \( R \) given by \( I = \bigcap \mathcal{F} \). Thus the set \( \mathcal{F} \subseteq \varnothing \) is closed in \( \varnothing \) if and only if \( \mathcal{F} = h(I) \) for some ideal \( I \) in \( R \). Further the space \( \varnothing \) is a \( T_0 \)-space.

The maximal spectrum of \( R \) is the set \( \mathcal{M} \) topologised by assuming the base for open sets the family \( \mathcal{B}_1 = \{ U(a) \mid a \in R \} \), where \( U(a) = \{ M \in \mathcal{M} \mid a \notin M \} \). Obviously, \( \mathcal{M} \) is a subspace of the space \( \varnothing \). Note that \( U(a) = \emptyset \iff a \in E \).

As \( R \) contains maximal elements, \( \mathcal{M} \) is a compact space. Further for each \( M \in \mathcal{M}, \{ M \} \) being a closed set in \( \mathcal{M} \), the space \( \mathcal{M} \) is a \( T_1 \)-space. The set \( \mathcal{M} \) is the set of \( T_1 \)-points in \( \varnothing \).

Let \( \Sigma = \{ P \in \varnothing \mid \bigcap \mathcal{M} \subseteq P \} \subseteq \varnothing \). i.e the set \( \Sigma \) is the set of all prime ideals consisting of all dual dense elements in \( R \). Thus \( \mathcal{M} \subseteq \Sigma \subseteq \varnothing \). \( \Sigma \) endowed with the hull kernel topology forms a subspace of the space \( \varnothing \). The base for open sets is the family \( \mathcal{B}_2 = \{ X(a) \mid a \in R \} \), where \( X(a) = \{ P \in \Sigma \mid a \notin P \} \). Obviously, \( \mathcal{M} \) is a subspace of the space \( \Sigma \) and \( \Sigma \) is a subspace of the space \( \varnothing \).

First of all we prove a result which is crucial to our approach besides being of independent interest

Lemma 3.1. Let each \( P \in \Sigma \) be contained in a unique maximal ideal. Define \( g : \Sigma \to \mathcal{M} \) by \( g(P) = M \) where \( M \) is the unique maximal ideal containing \( P \). Then \( g \) is a continuous map.
Proof. The map \( g : \Sigma \rightarrow \mathcal{M} \) is a well defined map. Select \( a \in R \). To prove that \( g \) is continuous, it is enough to prove \( g^{-1}[Y(a)] \) is closed in \( \Sigma \). Define 
\[
I = \bigcap g^{-1}[Y(a)] \quad \text{and} \quad B = \bigcup g^{-1}[Y(a)].
\]
Select \( P_1 \in \text{cl}_\Sigma g(Y(a)) = \{P \in \Sigma \mid I \subseteq P \} \). For any \( t \notin P_1 \), there exists \( P_2 \in \Sigma \) such that \( a \in g(P_2) \) and \( t \notin P_2 \) as \( t \notin I \). Hence for any \( s \in B, s \wedge t \notin P_2 \) and hence \( s \wedge t \notin I \). But this gives \([(R \setminus B) \wedge (R \setminus I)] \cap I = \emptyset \). Hence by result (2.6) there exists a prime ideal \( Q \) in \( R \) such that \( I \subseteq Q \) and \([(R \setminus B) \wedge (R \setminus I)] \cap Q = \emptyset \). Let if possible \( q \vee a \) is a maximal element in \( R \) for some \( q \in Q \) i.e. \( q \in (a)^+ \). As \( Q \subseteq B, q \in B \). Hence there exists some \( P_3 \in \Sigma \) such that \( q \in P_3 \) and \( a \in g(P_3) \). Thus \( q \vee a \in g(P_3) \), which is absurd. Hence \( q \vee a \) is not a maximal elements in \( R \) for any \( q \in Q \). This shows that \( Q \vee (a) \) is a proper ideal and hence contained in some maximal ideal say \( M \) in \( R \). As \( P_1 \subseteq Q \subseteq M \), by assumption, \( g(P_1) = g(Q) = M \). But \( a \in M \) implies \( a \in g(P_1) \) i.e. \( P_1 \in g^{-1}[Y(a)] \). Thus \( P_1 \in \text{cl}_\Sigma g^{-1}[Y(a)] \) implies \( P_1 \in g^{-1}[Y(a)] \). Hence \( \text{cl}_\Sigma g^{-1}[Y(a)] = g^{-1}[Y(a)] \). This in turn shows that \( g \) is a continuous map. \( \blacksquare \)

For any \( M \in \mathcal{M} \), define \( \Sigma_M = \{P \in \Sigma \mid P \subseteq M \} \). By the continuity of the function \( g \) (defined in Lemma 3.1) we get the following properties of \( \Sigma_M \).

**Theorem 3.2.** Following statements are equivalent in \( R \).

1. Each \( P \in \Sigma \) is contained in a unique maximal ideal in \( R \).

2. For any \( M \in \mathcal{M} \), \( \Sigma_M \) is a closed set in \( \Sigma \).

3. \( M \) is the unique maximal ideal containing \( \cap \Sigma_M \)

Proof. (1) \( \Rightarrow \) (2)

By Lemma 3.1, the map \( g : \Sigma \rightarrow \mathcal{M} \) defined by \( g(P) = M \), where \( M \) is the maximal ideal in \( R \), is continuous. Further \( \{M\} \) being closed in \( \mathcal{M} \) we get \( g^{-1}(\{M\}) = \{P \in \Sigma_M \mid P \subseteq M \} = \Sigma_M \) is closed in \( \Sigma \) and the implication follows.

(2) \( \Rightarrow \) (3)

Let \( M_1 \in \mathcal{M} \) such that \( \cap \Sigma_M \subseteq M_1 \). Then \( M_1 \in \text{cl}_\Sigma \Sigma_M \) implies \( M_1 \in \Sigma_M \) by assumption. But then \( M_1 \subseteq M \). This is possible only when \( M_1 = M \). Hence \( M \) is the unique maximal ideal containing \( \cap \Sigma_M \).

(3) \( \Rightarrow \) (1)

Let \( P \in \Sigma \) be such that \( P \subseteq M_1 \) and \( P \subseteq M_2 \) for \( M_1, M_2 \in \mathcal{M} \). \( P \subseteq M_1 \) implies \( P \in \Sigma_{M_1} \) and hence \( \cap \Sigma_{M_1} \subseteq P \). But then as \( P \subseteq M_2 \), \( \cap \Sigma_{M_1} \subseteq M_2 \).

By the condition (3) \( M_1 = M_2 \). Hence the implication follows.

Thus the statements (1), (2) and (3) are equivalent. \( \blacksquare \)

The retraction of the subspace \( \mathcal{M} \) of \( \Sigma \) characterizes the algebraic properties of \( R \). This we prove in the following theorem.
Theorem 3.3. \( M \) is a retract of \( \Sigma \) iff each \( P \in \Sigma \) is contained in a unique maximal ideal in \( R \).

Proof. Let \( M \) be a retract of \( \Sigma \). Hence there exists a continuous map \( \gamma : \Sigma \to M \) such that \( \gamma(M) = M \), for all \( M \in M \). Select \( P \in \Sigma \) and let \( \gamma(P) = M \). \( \{M\} \) being closed in \( M \), \( \gamma^{-1}(\{M\}) \) is closed in \( \Sigma \). As \( P \in \gamma^{-1}(\{M\}) \), \( cl_{\Sigma}\{P\} \subseteq \gamma^{-1}(\{M\}) \). If \( P \subseteq M_1 \) for some \( M_1 \in M \), then \( M_1 \in \gamma^{-1}(\{P\}) \) and hence \( \gamma(M_1) = M \). This shows that if \( \gamma(P) = M_1 \), then \( P \subseteq M \) and \( M \) is the unique maximal ideal containing \( P \).

Conversely, let each \( P \in \Sigma \) is contained in a unique maximal ideal in \( R \). Then the map \( g : \Sigma \to M \) defined in Lemma 3.1, is a retraction of \( \Sigma \) on to \( M \). Hence \( M \) is retract of \( \Sigma \).

Corollary 3.4. If each \( P \in \Sigma \) is contained in a unique maximal ideal in \( R \), then the map \( g : \Sigma \to M \) defined in Lemma 3.1, is the unique retraction of \( \Sigma \) on to \( M \).

Proof. Let \( \gamma : \Sigma \to M \) be a retraction and \( \gamma(P) = M \) for \( P \in \Sigma \). Then \( M \) will be the unique maximal ideal containing \( P \). (see the proof of only if part of theorem 3.3). Hence \( \gamma(P) = g(P) \). This is true for any \( P \in \Sigma \), \( \gamma = g \). Hence \( g \) is the unique retraction of \( \Sigma \) on to \( M \).

In the following theorem we prove some necessary and sufficient conditions for the maximal spectrum \( M \) to be Hausdorff.

Theorem 3.5. Following statements are equivalent in \( R \).

1. Each \( P \in \Sigma \) is contained in a unique maximal ideal in \( R \).
2. Any two distinct minimal prime filters in \( R \) are weakly co-maximal.
3. For \( M_1 \neq M_2 \) in \( M \), there exist \( a_1 \neq M_1 \), \( a_2 \neq M_2 \) such that \( a_1 \land a_2 \in E \).
4. \( M \) is a \( T_2 \) space.
5. \( M \) is a normal.

Proof. 1) \( \Rightarrow \) 2)
Suppose there exist two distinct prime filters \( F_1 \) and \( F_2 \) in \( R \) such that \( (F_1 \lor F_2) \cap E = \phi \). Then \( F_1 \lor F_2 \) being a proper filter, it must be contained in some ultra filter say \( U \) in \( R \). But then the prime ideal \( R \setminus U \) is contained in two distinct maximal ideals \( R \setminus F_1 \) and \( R \setminus F_2 \) of \( R \). This contradicts our assumption. Hence 1) \( \Rightarrow \) 2).

2) \( \Rightarrow \) 3)
Let \( M_1 \neq M_2 \) in \( M \). Then \( R \setminus M_1 \) and \( R \setminus M_2 \) are distinct minimal prime filters in \( R \). Hence by assumption \( [(R \setminus M_1) \lor (R \setminus M_2)] \cap E = \phi \). Hence there
exist $a_1 \notin M_1$, $a_2 \notin M_2$ such that $a_1 \land a_2 \in E$.

3) $\Rightarrow$ 4)

Let $M_1 \neq M_2$ in $\mathcal{M}$. by assumption there exist $a_1 \notin M_1$, $a_2 \notin M_2$ such that $a_1 \land a_2 \in E$. As $E = \cap \mathcal{M}$, $U(a_1 \land a_2) = U(a_1) \cap U(a_2) = \phi$. Thus for $M_1 \neq M_2$ in $\mathcal{M}$, there exist disjoint basic open sets $U(a_1)$ and $U(a_2)$ in $\mathcal{M}$ containing $M_1$ and $M_2$ respectively. Hence $\mathcal{M}$ is a $T_2$ space.

4) $\Rightarrow$ 1)

Let $P \in \Sigma$ be such that $P \subseteq M_1$ and $P \subseteq M_2$ for $M_1 \neq M_2$ in $\mathcal{M}$. As $\mathcal{M}$ is a $T_2$ space, there exist basic open sets $U(a)$ and $U(b)$ such that $M_1 \in U(a)$, $M_2 \in U(b)$ and $U(a) \cap U(b) = \phi$. But then $U(a \land b) = \phi$ implies $a \land b \in E$. As $E \subseteq P$ and $P$ is prime, either $a \in P$ or $b \in P$. But then $a \in M_1$ or $b \in M_2$; which is absurd. Hence each $P \in \Sigma$ must be contained in a unique maximal ideal.

(4) $\Rightarrow$ (5)

$\mathcal{M}$ is a normal space as $\mathcal{M}$ is a compact, Hausdorff space.

(5) $\Rightarrow$ (4)

Let $\mathcal{M}$ be a normal space. As $\mathcal{M}$ is always $T_1$ space we get $\mathcal{M}$ is a $T_3$ space and hence $\mathcal{M}$ is a $T_2$-space.

Thus (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) $\iff$ (5) and hence the result follows. 

We know that the members of $\mathcal{M}$ are the $T_1$ points of $\Sigma$. Hence they can be separated by neighbourhoods in $\Sigma$. But if they are separated by disjoint neighbourhoods in $\Sigma$, then we have

**Theorem 3.6.** The following statements are equivalent in $R$.

1. Each $P \in \Sigma$ is contained in a unique maximal ideal in $R$.

2. $\Sigma$ is normal.

3. Any two distinct elements in $\mathcal{M}$ can be separated by disjoint neighbourhoods in $\Sigma$.

*Proof.* (1) $\Rightarrow$ (2)

Let $P \in \Sigma$ be contained in the unique maximal ideal in $R$. Then by Lemma 3.1, the mapping $g : \Sigma \rightarrow \mathcal{M}$ defined by $g(P) = M$, where $M$ is the unique maximal ideal, is continuous. As $\Sigma$ is a compact space and $\mathcal{M}$ is a $T_2$-space we get the mapping $g : \Sigma \rightarrow \mathcal{M}$ is a closed mapping. Let $F_1$ and $F_2$ be any two disjoint closed sets in $\Sigma$. Then $g(F_1)$ and $g(F_2)$ will be disjoint closed sets in $\mathcal{M}$. $\mathcal{M}$ being normal, there exist disjoint neighbourhood say $N_1$ and $N_2$ of $g(F_1)$ and $g(F_2)$ respectively in $\mathcal{M}$. But then $g^{-1}(N_1)$ and $g^{-1}(N_2)$ will be disjoint neighbourhoods of $F_1$ and $F_2$ respectively in $\Sigma$. This shows that the space $\Sigma$ is normal.

2) $\Rightarrow$ 3)
Let $M_1 \neq M_2$ in $\mathfrak{M}$. Then $\{M_1\}$ and $\{M_2\}$ will be disjoint closed sets in $\Sigma$. $\Sigma$ being a normal space, the implication follows.

3) $\Rightarrow$ 1) Let there exist $P \in \Sigma$ such that $P \subseteq M_1$ and $P \subseteq M_2$ for $M_1 \neq M_2$ in $\mathfrak{M}$. By assumption, there exist disjoint basic open sets say $X(a)$ and $X(b)$ containing $M_1$ and $M_2$ respectively in $\Sigma$. As $X(a) \cap X(b) = X(a \wedge b) = \emptyset$ we get $a \wedge b \in E$. As $P \in \Sigma$, $a \wedge b \in P$. Thus either $a \in P$ or $b \in P$. But then either $M_1 \notin X(a)$ or $M_2 \notin X(b)$ which is not true. Hence the implication.

As (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1), the result follows.

If 0 is the only dual dense elements in $R$, i.e. if $E = \cap \mathfrak{M} = \{0\}$, then $R$ is dually semi-complemented ADL and conversely [3]. In this case $\Sigma = \emptyset$ and therefore from all the previous results as a special case we have

**Theorem 3.7.** Let $R$ be a dually semi-complemented ADL. The following statements are equivalent

1. Each $P \in \emptyset$ is contained in a unique maximal ideal in $R$. i.e. $R$ is a pm-ADL.

2. $\mathfrak{M}$ is normal.

3. $\emptyset$ is normal.

4. For each $M \in \mathfrak{M}$, the set $\wp_M = \{P \in \emptyset \mid P \subseteq M\}$ is a closed set.

5. $M$ is the unique maximal ideal containing $\cap \wp_M$

6. $M$ is the unique maximal ideal containing $O(M)$, where $O(M) = \{x \in R \mid \text{there exists } y \notin M \text{ such that } x \wedge y = 0\}$.

7. $\mathfrak{M}$ is a retract of $\emptyset$.

8. Any two distinct minimal prime filters in $R$ are co-maximal.

9. For $M_1 \neq M_2$ in $\mathfrak{M}$, there exist $a_1 \notin M_1$ and $a_2 \notin M_2$ such that $a_1 \wedge a_2 = 0$.

10. Any two distinct members of $\mathfrak{M}$ can be separated by disjoint neighbourhoods in the $\Sigma$.

**Theorem 3.8.** $\emptyset$ is a regular space if and only if for any $P \in \emptyset$ and $a \notin P$, for $a \in R$. There exists an ideal $I$ of $R$ and $b \in R$ such that $P \in V(b) \subseteq h(I) \subseteq V(a)$.
Proof. Let $\phi$ be a regular space. Let $P \in \phi$ and $a \notin P$ for some $a \in R$. Then $P \notin \phi \setminus V(a)$. As $\phi$ is a regular space, there exist two disjoint open sets $G$ and $H$ in $\phi$ such that $P \in G$ and $\phi \setminus V(a) \subseteq H$. Therefore $\phi \setminus H \subseteq V(a)$. But $\phi \setminus H = h(I)$ for some ideal $I$ in $R$. Thus $\phi \setminus H = h(I) \subseteq V(a)$. $G \cap H = \phi$ will imply $H \subseteq \phi \setminus G$. $\phi \setminus G$ being a closed set, $\phi \setminus G = h(J)$ for some ideal $J$ in $R$. Then as $H \subseteq h(J)$, $P \in G \Rightarrow P \notin \phi \setminus G = h(J)$ and hence $J \notin P$. Select $b \in J$ such that $b \notin P$. But then $P \in V(b)$. Let $T \in H \subseteq h(J)$. Then $J \subseteq T$. As $b \in J \Rightarrow b \in T$ it follows that $T \in h(b)$. This shows that $H \subseteq h(b)$. Hence $\phi \setminus h(b) \subseteq \phi \setminus H = h(I)$. i.e. $V(b) \subseteq h(I)$. Thus for any $P \in \phi$ and $a \notin P$, there exist an ideal $I$ in $R$ such that $P \in V(b) \subseteq h(I) \subseteq V(a)$.

Conversely, suppose that for any $P \in \phi$ and $a \notin P$, for $a \in R$. There exists an ideal $I$ of $R$ and $b \in R$ such that $P \in V(b) \subseteq h(I) \subseteq V(a)$. To show that the space $\phi$ is regular. Let $P \in \phi$ and $h(K)$ be any closed set of $\phi$ such that $P \notin h(K)$. This gives $K \notin P$. Therefore there exist $a \in K$ such that $a \notin P$. $a \notin P \Rightarrow P \in V(a)$. As $a \in K \subseteq R$ and $a \notin P$ by assumption there exist an ideal $I$ of $R$ such that $P \in V(b) \subseteq h(I) \subseteq V(a)$. We have $V(a) \cap h(K) = \phi$, since for $a \in K$, $K \in h(a)$. Therefore $h(K) \subseteq \phi \setminus V(a) \subseteq \phi \setminus h(I)$. Also $V(b) \cap (\phi \setminus h(I)) = \phi$. Thus there exist two disjoint open sets $V(b)$ and $(\phi \setminus h(I))$ such that $P \in V(b)$ and $h(K) \subseteq (\phi \setminus h(I))$. Therefore the space is regular. 

Corollary 3.9. If for any $P \in \Sigma$ and $a \notin P$, for $a \in R$. There exists an ideal $I$ of $R$ and $b \in R$ such that $P \in X(b) \subseteq h(I) \subseteq X(a)$, then $\Sigma$ is a $T_3$-space.

Corollary 3.10. If for any $P \in \mathcal{M}$ and $a \notin P$, for $a \in R$. There exists an ideal $I$ of $R$ and $b \in R$ such that $P \in U(b) \subseteq h(I) \subseteq U(a)$, then $\mathcal{M}$ is a $T_3$-space.

References


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