Some Conditions for Strongly Starlikeness of Certain Analytic Functions

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Abstract
For analytic functions $f(z)$ in the open unit disk $U$ with $f(0) = 0$ and $f'(0) = 1$, S. S. Miller, P. T. Mocanu and M. O. Reade (Proc. Amer. Math. Soc. 37(1973)) have introduced $\alpha$-convex functions in $U$ with some condition, and Z. Lewandowski, S. S. Miller and E. Zlotkiewicz (Ann. Univ. Marie-Curie Sklodowska 27(1974)) have discussed $\gamma$-starlike functions in $U$ with some condition. The object of the
present paper is to consider some sufficient conditions for strongly starlike of order
\( \beta \) for \( f(z) \in \mathcal{A} \).

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order \( \beta \)

1 Introduction

Let \( \mathcal{A} \) be the class of functions \( f(z) \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). For \( f(z) \in \mathcal{A} \), we say
that \( f(z) \) is \( \alpha \)-convex (or \( \alpha \)-starlike) in \( \mathbb{D} \) if \( f(z) \) satisfies

\[
\left( 1 - \alpha \right) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \neq 0 \quad (z \in \mathbb{D})
\]

for some real number \( \alpha \). For such \( \alpha \)-convex functions, Miller, Mocanu and Reade [2] have
shown that all \( \alpha \)-convex functions are univalent and starlike in \( \mathbb{D} \) for all \( \alpha \) \((-\infty \leq \alpha \leq \infty)\).

Further, we say that \( f(z) \in \mathcal{A} \) is \( \gamma \)-starlike in \( \mathbb{D} \) if \( f(z) \) satisfies

\[
f(z) f'(z) \left( 1 + \frac{zf''(z)}{f'(z)} \right) \neq 0 \quad (0 < |z| < 1) \quad \text{and}
\]

\[
\text{Re} \left\{ \left( \frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\gamma \right\} > 0 \quad (z \in \mathbb{D})
\]

for some real number \( \gamma \), where the powers appearing in (1.3) are meant as principal values.

For such \( \gamma \)-starlike functions, Lewandowski, Miller and Zlotkiewicz [1] have proved
that all \( \gamma \)-starlike functions are univalent and starlike for all \( \gamma \) \((-\infty \leq \gamma \leq \infty)\). If
\( f(z) \in \mathcal{A} \) satisfies

\[
|\arg \frac{zf'(z)}{f(z)}| < \frac{\pi}{2} \beta \quad (z \in \mathbb{D})
\]
Strongly starlikeness of analytic functions

for some real number $\beta$ $(0 < \beta \leq 1)$, then $f(z)$ is said to be strongly starlike of order $\beta$ in $\mathbb{U}$. We denote by $S(\beta)$ all strongly starlike functions of order $\beta$ in $\mathbb{U}$. We note that $S(\beta_1) \subset S(\beta_2)$ for $0 < \beta_1 \leq \beta_2 \leq 1$ and that $f(z) \in S(1)$ if and only if

\begin{equation}
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}).
\end{equation}

Thus, the class $S(\beta)$ is the subclass of all starlike functions in $\mathbb{U}$.

In the present paper, we investigate some sufficient conditions for strongly starlikeness of order $\beta$ of certain $\alpha$-convex functions or of certain $\gamma$-starlike functions.

2 Sufficient conditions for strongly starlikeness

In order to discuss our problems, we need the following lemma due to Nunokawa [3].

**Lemma 1** Let a function $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \not= 0$ ($z \in \mathbb{U}$). If there exists a point $z_0 \in \mathbb{U}$ such that

\[ |\arg(p(z))| < \frac{\pi}{2} \alpha \quad (|z| < |z_0|) \]

and

\[ |\arg(p(z_0))| = \frac{\pi}{2} \alpha \]

for some $\alpha > 0$, then

\begin{equation}
\frac{z_0p'(z_0)}{p(z_0)} = ik\alpha,
\end{equation}

where

\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{(when } \arg(p(z_0)) = \frac{\pi}{2} \alpha) \]

and

\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{(when } \arg(p(z_0)) = -\frac{\pi}{2} \alpha) \],

where

\[ p(z_0)^{\frac{1}{a}} = \pm ia \quad (a > 0). \]
Applying the above lemma, we derive

**Theorem 1** If \( f(z) \in \mathcal{A} \) satisfies

\[
(2.2) \quad \left| \arg \left\{ \left( \frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\gamma \right\} \right| < \frac{\pi}{2} \delta \quad (z \in \mathbb{U})
\]

with \( \gamma > 0 \) and

\[
(2.3) \quad \delta = \beta + \frac{2\gamma}{\pi} \tan^{-1} \left( \frac{\beta \sin \left( \frac{\pi}{2} (1 - \beta) \right)}{(1 + \beta) \frac{1+\beta}{2} (1 - \beta) \frac{1+\beta}{2} + \beta \cos \left( \frac{\pi}{2} (1 - \beta) \right)} \right),
\]

where \( 0 < \beta < 1 \), then \( f(z) \in \mathcal{S}(\beta) \).

**Proof.** We note that the argument at the origin and the argument at the point at infinity can not be defined. Therefore \( f(z) \neq 0 \) for \( 0 < |z| < 1 \) and \( f'(z) \neq 0 \) for \( z \in \mathbb{U} \) with the condition (2.2). Thus we define the function \( p(z) \) by

\[
p(z) = \frac{zf'(z)}{f(z)} \quad (f(z) \in \mathcal{A}).
\]

Then \( p(z) \) is analytic in \( \mathbb{U} \) and \( p(0) = 1 \). It follows that

\[
\left( \frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\gamma = p(z)^{1-\gamma} \left( p(z) + \frac{zp'(z)}{p(z)} \right)^\gamma
\]

\[
= p(z) \left( 1 + \frac{zp'(z)}{p(z)^2} \right)^\gamma
\]

and that

\[
\left| \arg(p(z)) + \gamma \arg \left( 1 + \frac{zp'(z)}{p(z)^2} \right) \right| < \frac{\pi}{2} \delta \quad (z \in \mathbb{U}).
\]

If there exists a point \( z_0 \in \mathbb{U} \) such that

\[
|\arg(p(z))| < \frac{\pi}{2} \beta \quad (|z| < |z_0|)
\]

and

\[
|\arg(p(z_0))| = \frac{\pi}{2} \beta,
\]

then Lemma 1 gives us that

\[
\frac{z_0p'(z_0)}{p(z_0)} = ik\beta.
\]
If \( \arg(p(z_0)) = \frac{\pi}{2} \beta \), then we have \( p(z_0)^\frac{1}{2} = ia \) \((a > 0)\). Therefore, we see that

\[
p(z_0) \left( 1 + \frac{z_0 p'(z_0)}{p(z_0)^2} \right)^\gamma = a^\beta e^{i\frac{\pi}{2} \beta} \left( 1 + \frac{k\beta}{a^\beta} e^{i\frac{\pi}{2} (1-\beta)} \right)^\gamma
\]

with

\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right).
\]

Since

\[
\frac{k\beta}{a^\beta} \geq \frac{\beta}{2} (a^{1-\beta} + a^{-1-\beta})
\]

\( g(a) = \frac{1}{2} (a^{1-\beta} + a^{-1-\beta}) \) \((a > 0)\) takes the minimum value at \( a = \left( \frac{1+\beta}{1-\beta} \right)^{\frac{1}{2}} \) because

\[
g'(a) = \frac{1}{2} \left( 1 - \beta \right) \frac{1}{a^{2+\beta}} - \frac{1 + \beta}{a^{2+\beta}} = -\beta \frac{1 + \beta}{2a^{2+\beta}} (a^2 - \frac{1 + \beta}{1 - \beta}).
\]

Thus we have

\[
\arg(p(z_0)) + \gamma \arg \left( 1 + \frac{z_0 p'(z_0)}{p(z_0)^2} \right) = \frac{\pi}{2} \beta + \gamma \arg \left( 1 + \frac{k\beta}{a^\beta} e^{i\frac{\pi}{2} (1-\beta)} \right)
\]

\[
\geq \frac{\pi}{2} \beta + \gamma \tan^{-1} \left( \frac{\beta \sin \left( \frac{\pi}{2} (1 - \beta) \right)}{(1 + \beta)^{\frac{1+\beta}{2}} (1 - \beta)^{\frac{1-\beta}{2}} + \beta \cos \left( \frac{\pi}{2} (1 - \beta) \right)} \right)
\]

\[
= \frac{\pi}{2} \delta,
\]

which contradicts the condition (2.2).

If \( \arg(p(z_0)) = -\frac{\pi}{2} \beta \), applying the same method, we also have

\[
\arg(p(z_0)) + \gamma \arg \left( 1 + \frac{z_0 p'(z_0)}{p(z_0)^2} \right) \leq -\frac{\pi}{2} \beta - \gamma \tan^{-1} \left( \frac{\beta \sin \left( \frac{\pi}{2} (1 - \beta) \right)}{(1 + \beta)^{\frac{1+\beta}{2}} (1 - \beta)^{\frac{1-\beta}{2}} + \beta \cos \left( \frac{\pi}{2} (1 - \beta) \right)} \right)
\]

\[
= -\frac{\pi}{2} \delta,
\]

which contradicts the condition (2.2). This implies that there is no point \( z_0 \in U \) such that

\[
|\arg(p(z_0))| = \left| \arg \left( \frac{z_0 f'(z_0)}{f(z_0)} \right) \right| = \frac{\pi}{2} \beta.
\]

This completes the proof of the theorem. \( \square \)
Next, we prove the following theorem for $\beta = 1$.

**Theorem 2** If $f(z) \in \mathcal{A}$ satisfies

\[
\left| \arg \left\{ \left( \frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{\gamma} \right\} \right| < \frac{\pi}{2} \quad (z \in \mathbb{U})
\]

with $\gamma \neq 0$, then $f(z) \in S(1)$ or

\[\Re \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}).\]

**Proof.** With the comment for the arguments at the origin and at the point at infinity in the proof of Theorem 1, we consider the function $p(z)$ defined by

\[p(z) = \frac{zf'(z)}{f(z)},\]

then $p(z)$ is analytic in $\mathbb{U}$, $p(0) = 1$ and $p(z) \neq 0$ ($z \in \mathbb{U}$). If there exists a point $z_0 \in \mathbb{U}$ such that

\[|\arg(p(z_0))| < \frac{\pi}{2} \quad (|z| < |z_0|)\]

and

\[|\arg(p(z_0))| = \frac{\pi}{2},\]

then Lemma 1 shows us that

\[
z_0p'(z_0) \quad p(z_0) = ik.
\]

If $\arg(p(z_0)) = \frac{\pi}{2}$, then we know that

\[
\arg \left\{ \left( \frac{z_0f'(z_0)}{f(z_0)} \right)^{1-\gamma} \left( 1 + \frac{z_0f''(z_0)}{f'(z_0)} \right)^{\gamma} \right\} = \arg(p(z_0)) + \gamma \arg \left( 1 + \frac{z_0p'(z_0)}{p(z_0)^2} \right)
\]

\[= \frac{\pi}{2} + \gamma \arg \left( 1 + \frac{ik}{ia} \right) = \frac{\pi}{2},\]

where $p(z_0) = ia$ ($a > 0$) and $k \geq \frac{1}{2} \left( a + \frac{1}{a} \right)$.

If $\arg(p(z_0)) = -\frac{\pi}{2}$, then we see that

\[
\arg \left\{ \left( \frac{z_0f'(z_0)}{f(z_0)} \right)^{1-\gamma} \left( 1 + \frac{z_0f''(z_0)}{f'(z_0)} \right)^{\gamma} \right\} = -\frac{\pi}{2} + \gamma \arg \left( 1 - \frac{ik}{ia} \right) = -\frac{\pi}{2}.
\]
where \( p(z_0) = -ia \ (a > 0) \) and \( k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \).

This contradicts our condition (2.4) of the theorem. Thus there is no \( z_0 \in \mathbb{U} \) such that

\[
|\arg(p(z))| < \frac{\pi}{2} \ (|z| < |z_0|)
\]

and

\[
|\arg(p(z_0))| = \frac{\pi}{2}.
\]

This completes the proof of the theorem. \( \square \)

**Remark 1** Theorem 2 was obtained by Lewandowski, Miller and Złotkiewicz [1] with the another method for the proof.

Furthermore, we discuss

**Theorem 3** If \( f(z) \in \mathcal{A} \) satisfies

\[
(2.5) \quad \left| \arg \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} \right| < \frac{\pi}{2} \rho \quad (z \in \mathbb{U})
\]

with \( \alpha > 0 \) and

\[
(2.6) \quad \rho = \beta + \frac{2}{\pi} \tan^{-1} \left( \frac{\alpha \beta \sin \left( \frac{\pi}{2} (1 - \beta) \right)}{(1 + \beta)\frac{1+\beta}{2} (1 - \beta)\frac{1-\beta}{2} + \alpha \beta \cos \left( \frac{\pi}{2} (1 - \beta) \right)} \right),
\]

where \( 0 < \beta < 1 \), then \( f(z) \in \mathcal{S}(\beta) \).

**Proof.** Defining the function \( p(z) \) by

\[
p(z) = \frac{zf'(z)}{f(z)},
\]

we see that \( p(z) \) is analytic in \( \mathbb{U} \) and \( p(0) = 1 \). It follows that

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) = p(z) \left( 1 + \alpha \frac{zp'(z)}{p(z)^2} \right),
\]

and that

\[
\left| \arg(p(z)) + \arg \left( 1 + \alpha \frac{zp'(z)}{p(z)^2} \right) \right| < \frac{\pi}{2} \rho \quad (z \in \mathbb{U}).
\]
We suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg(p(z))| < \frac{\pi}{2} \beta \quad (|z| < |z_0|)$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2} \beta.$$ 

Then, using Lemma 1, we have that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta.$$ 

Therefore, if $\arg(p(z_0)) = \frac{\pi}{2} \beta$, then we see that

$$\arg(p(z_0)) + \arg\left(1 + \alpha \frac{z_0 p'(z_0)}{p(z_0)^2}\right) = \frac{\pi}{2} \beta + \arg\left(1 + \alpha \frac{k\beta}{a^2} e^{i\frac{\pi}{2}(1-\beta)}\right)$$

$$\geq \frac{\pi}{2} \beta + \tan^{-1}\left(\frac{\alpha \beta \sin\left(\frac{\pi}{2}(1-\beta)\right)}{(1 + \beta) \frac{1+\beta}{1-\beta} (1 - \beta)^{-\frac{1+\beta}{1-\beta}} + \alpha \beta \cos\left(\frac{\pi}{2}(1 - \beta)\right)}\right)$$

$$= \frac{\pi}{2} \rho,$$

which contradicts the condition (2.5).

If $\arg(p(z_0)) = -\frac{\pi}{2} \beta$, then, applying the same method, we obtain that

$$\arg(p(z_0)) + \arg\left(1 + \alpha \frac{z_0 p'(z_0)}{p(z_0)^2}\right) = -\frac{\pi}{2} \beta + \arg\left(1 - \alpha \frac{k\beta}{a^2} e^{-i\frac{\pi}{2}(1-\beta)}\right)$$

$$\leq -\frac{\pi}{2} \beta - \tan^{-1}\left(\frac{\alpha \beta \sin\left(\frac{\pi}{2}(1-\beta)\right)}{(1 + \beta) \frac{1+\beta}{1-\beta} (1 - \beta)^{-\frac{1+\beta}{1-\beta}} + \alpha \beta \cos\left(\frac{\pi}{2}(1 - \beta)\right)}\right)$$

$$= -\frac{\pi}{2} \rho,$$

which contradicts our condition (2.5). Therefore, we conclude that $f(z) \in S(\beta)$. 

\begin{proof}
\end{proof}

3 Appendix

Let us consider the function

$$p(z) = \frac{1}{1 - z^{-\frac{1}{n}}}$$
with \( \alpha < 0 \). Then we have that

\[
p(z) + \alpha \frac{zp'(z)}{p(z)} = \frac{1}{1 - z^{-\alpha}} - \alpha z \frac{1}{1 - z^{-\alpha}} = 1,
\]

or

\[
\arg \left( p(z) + \alpha \frac{zp'(z)}{p(z)} \right) \equiv 0 \quad (z \in \mathbb{U}).
\]

This implies that

\[
\left| \arg \left( p(z) + \alpha \frac{zp'(z)}{p(z)} \right) \right| \equiv 0 \quad (z \in \mathbb{U}).
\]

But we see that

\[
\lim_{r \to 1^-} \left\{ \max_{|z|=r<1} |\arg(p(z))| \right\} = \lim_{r \to 1^-} \left\{ \sin^{-1}\left( r^{-\frac{1}{\alpha}} \right) \right\} = \frac{\pi}{2}.
\]

If we take

\[
p(z) = \frac{zf'(z)}{f(z)},
\]

then \( p(z) \) satisfies \( p(0) = 1 \) and

\[
p(z) + \alpha \frac{zp'(z)}{p(z)} = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right),
\]

or

\[
\arg \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} \equiv 0 \quad (z \in \mathbb{U}).
\]

Therefore, we can not consider such a sufficient condition

\[
\left| \arg \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} \right| < \frac{\pi}{2} \quad (z \in \mathbb{U})
\]

for \( \alpha < 0 \) and for a sufficiently small and positive real number \( \varepsilon \) to be \( f(z) \in S(\rho) \) with a sufficiently small and positive real number \( \rho \).

References


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