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On Controllability of Some Stochastic Semilinear Integrodifferential Systems in Hilbert Spaces

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Abstract

In this paper, approximate and complete controllability for semilinear stochastic integrodifferential systems are considered. The results are obtained by using the Banach fixed point theorem. An example is provided to illustrate the theory.

Keywords: Complete and approximate controllability, Stochastic integrodifferential equations, Banach fixed point

1 Introduction

The problem of controllability of linear and nonlinear systems have been extensively studied by several authors (see [1]-[11] and references therein). Among the various methods which considered the controllability problem of infinite dimensional systems, the fixed point principles have been used for these systems. In view of such methods, the controllability problem is transformed into fixed point problem for an appropriate nonlinear operator equation in a function space. Mahmudov [9] established sufficient conditions for approximate controllability of nonlinear systems in Hilbert spaces by using the Nussbaum's fixed point theorem. The classical Banach fixed point theorem is used for the same purpose by many authors (see for examples [3],[10], and [11]). Motivated by these works, we use the Banach fixed point theorem to get a unique solution of nonlinear stochastic integrodifferential system in Hilbert spaces and then obtain sufficient conditions for controllability of such system.

2 Preliminaries

We are given a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ together with a normal filtration $\mathfrak{F}_t$, $t \geq 0$. We consider the separable Hilbert spaces X, U, and E, and w is

a Q-Wiener process on $(\Omega, \mathfrak{F}, \mathbf{P})$ with the linear bounded covariance operator Q such that $\mathrm{tr}Q < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}$ in E, a bounded sequence of nonnegative real numbers $\{\lambda_k\}$ such that $Qe_k = \lambda_k e_k$, $k = 1, 2, , ...$ and a sequence $\{\beta_k\}$ of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \langle e_k, e\rangle\, \beta_k(t), \qquad e \in E, t \in I = [0, T],$$

and $\mathfrak{F}_t = \mathfrak{F}_t^w$, where $\mathfrak{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \le s \le t\}$. Let $L_2^0 = L_2(Q^{1/2}E; X)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}E$ to X with the inner product $\langle \psi, \phi\rangle_{L_2^0}$ =tr $[\psi Q \phi^\star]$. $L_2(\mathfrak{F}_T; X)$ is a Hilbert space of all $\mathfrak{F}_T$-measurable square integrable random variables with values in X. $L_2^{\mathfrak{F}}(I; X)$ is the Hilbert space of all square integrable and $\mathfrak{F}_t$-adapted processes with values in X. We recall that h is said to be $\mathfrak{F}_t$-adapted if $h(t, \cdot) : \Omega \to X$ is $\mathfrak{F}_t$-measurable, a.e. $t \in I$. Let $C(I, L_2(\Omega, \mathfrak{F}, \mathbf{P}; X))$ be the Banach space of continuous maps from I into $L_2(\Omega, \mathfrak{F}, \mathbf{P}; X)$ satisfying the condition $\sup_{t\in I} \mathbf{E}\, \|x(t)\|^2 < \infty$. $\mathcal{X}$ $(\mathcal{U})$ is the closed subspace of $C(I, L_2(\Omega, \mathfrak{F}, \mathbf{P}; X))$ $(C(I, L_2(\Omega, \mathfrak{F}, \mathbf{P}; U)))$ consisting of measurable and $\mathfrak{F}_t$-adapted X-valued (U-valued) processes $\phi(\cdot) \in C(I, L_2(\Omega, \mathfrak{F}, \mathbf{P}; X))$ $(\phi(\cdot) \in C(I, L_2(\Omega, \mathfrak{F}, \mathbf{P}; U)))$ endowed with the norm

$$\|\phi\|^2 = \sup_{t\in I} \mathbf{E}\, \|\phi(t)\|^2 .$$

Consider the following stochastic semilinear integrodifferential system

$$\begin{cases} dx(t) = [Ax(t) + Bu(t) + f(t, x(t), \displaystyle\int_0^t g(s, u(s))ds)]dt \\ \qquad\qquad + h(t, x(t), \displaystyle\int_0^t k(s, u(s))ds)dw(t), t \in I \\ x(0) = x_0 \end{cases} \tag{1}$$

where $A \in \mathcal{D}(A) \subset X \to X$, generates a strongly continuous semigroup of bounded linear operators $S(\cdot)$, B is a bounded linear operator from the Hilbert space U into X, w is a Q-Wiener process. The nonlinearities $f : I \times X \times U \to X$, $g : I \times U \to U$, $h : I \times X \times U \to L_2^0$, and $k : I \times U \to U$ are assumed to satisfy some conditions. For brevity, we assume

$$\begin{cases} Gu(t) = \displaystyle\int_0^t g(s, u(s))ds \text{ and } Ku(t) = \int_0^t k(s, u(s))ds, \\ F(x, u)(t) = f(t, x(t), Gu(t)) \text{ and } H(x, u)(t) = f(t, x(t), Ku(t)) \end{cases}$$

and we use throughout the sequel the common norm $\|\cdot\|$.

Definition 2.1 *An adapted process is said to be a mild solution of (1), if for all $t \in I$, it satisfies*

$$x(t;x_0,u) = x(t) = S(t)x_0 + \int_0^t S(t-s)[Bu(s) + F(x,u)(s)]ds + \int_0^t S(t-s)H(x,u)(s)dw(s) \quad (2)$$

where $u \in U_{ad} = \mathcal{U}$.

Definition 2.2 *System (1) is said to be approximately (completely) controllable on the interval I if $\overline{\mathcal{R}_T(\xi)} = L_2(\Omega, \mathfrak{F}_T; X)$ ($\mathcal{R}_T(\xi) = L_2(\Omega, \mathfrak{F}_T; X)$), where $\mathcal{R}_t(x_0) = \{x(t;x_0,u) : u \in L_2^{\mathfrak{F}}(I;U)\}$.*

The approximate controllability means that it is possible to steer the system from the initial point x_0 to within a distance $\epsilon > 0$ from all the final points in the state space $L_2(\Omega, \mathfrak{F}_T; X)$ at time T. But complete controllability is stronger, which means that all the points in $L_2(\Omega, \mathfrak{F}_T; X)$ can be reached from the point x_0 at time T.

In this article, the way for studying the approximate controllability of the system (1), is to consider the corresponding linear system

$$\begin{cases} dx(t) = [Ax(t) + Bu(t) + \widetilde{F}(t)]dt + \widetilde{H}(t)dw(t), t \in I \\ x(0) = x_0, \end{cases} \quad (3)$$

where $\widetilde{F}(t) = f(t, \int_0^t g(s)ds)$ and $\widetilde{H}(t) = h(t, \int_0^t k(s)ds)$. The mild solution of the linear system (3) is given by

$$x(t) = S(t)x_0 + \int_0^t S(t-s)[Bu(s) + \widetilde{F}(s)]ds + \int_0^t S(t-s)\widetilde{H}(s)dw(s). \quad (4)$$

The following hypotheses are assumed throughout the paper:

(A1) $(f,h) : I \times X \times U \to X \times L_2^0$ and $(g,k) : I \times U \to U \times U$ satisfy the Lipschitz condition, that is, there exists a positive constant C such that

$$\begin{cases} \|f(t,x_1,u_1) - f(t,x_2,u_2)\|^2 + \|h(t,x_1,u_1) - h(t,x_2,u_2)\|^2 \\ \leq C(\|x_1 - x_2\|^2 + \|u_1 - u_2\|^2) \\ \|g(t,u_1) - g(t,u_2)\|^2 + \|k(t,u_1) - k(t,u_2)\|^2 \leq C\,\|u_1 - u_2\|^2 \end{cases}$$

for each $t \in I$, $x_1, x_2 \in X$, and $u_1, u_2 \in U$.

(A2) (f,h) and (g,k) are continuous on $I \times X \times U$ and $I \times U$ respectively, and satisfy

$$\begin{cases} \|f(t,x,u)\|^2 + \|h(t,x,u)\|^2 \leq C\left(1 + \|x\|^2 + \|u\|^2\right) \\ \|g(t,u)\|^2 + \|k(t,u)\|^2 \leq C\left(1 + \|u\|^2\right) \end{cases}$$

for each $t \in I$, $x \in X$, and $u \in U$.

(A3) (f,h) and (g,k) are continuous on $I \times X \times U$ and $I \times U$ respectively, and satisfy

$$\begin{cases} \|f(t,x,u)\|^2 + \|h(t,x,u)\|^2 \leq C \\ \|g(t,u)\|^2 + \|k(t,u)\|^2 \leq C \end{cases}$$

for each $t \in I$, $x \in X$, and $u \in U$.

(A4) The operator $\alpha R(\alpha, \Gamma_t^T) = \alpha(\alpha I + \Gamma_t^T)^{-1}$ converges to zero operator in the strong operator topology as $\alpha \to 0^+$ for any $t \in I$, where Γ_t^T is the linear controllability operator on X that given by

$$\Gamma_t^T = \int_t^T S(T-s)BB^*S(T-s)ds,$$

and it satisfies $\left\|\alpha R(\alpha, \Gamma_t^T)\right\| \leq 1$, for any $t \in I$.

Remark 2.3 *In view of (A1)*

$$\begin{aligned}
&\mathbf{E}\left\|Gu_1(t) - Gu_2(t)\right\|^2 + \mathbf{E}\left\|Ku_1(t) - Ku_2(t)\right\|^2 \\
&= \mathbf{E}\left\|\int_0^t \left(g(s,u_1(s)) - g(s,u_2(s))\right) ds\right\|^2 + \mathbf{E}\left\|\int_0^t \left(k(s,u_1(s)) - k(s,u_2(s))\right) ds\right\|^2 \\
&\leq t\mathbf{E}\int_0^t \left(\left\|g(s,u_1(s)) - g(s,u_2(s))\right\|^2 + \left\|k(s,u_1(s)) - k(s,u_2(s))\right\|^2\right) ds \\
&\leq Ct\mathbf{E}\int_0^t \left\|u_1(s) - u_2(s)\right\|^2 ds \leq Ct^2 \left\|u_1 - u_2\right\|^2.
\end{aligned}$$

On the other hand, (A2) implies that

$$\begin{aligned}\mathbf{E}\left(\|Gu(t)\|^2+\|Ku(t)\|^2\right) &= \mathbf{E}\left\|\int_0^t g(s,u(s))ds\right\|^2+\mathbf{E}\left\|\int_0^t g(s,u(s))ds\right\|^2\\ &\le t\mathbf{E}\int_0^t\left(\|g(s,u(s))\|^2+\|k(s,u(s))\|^2\right)ds\\ &\le Ct\mathbf{E}\int_0^t\left(1+\|u(s)\|^2\right)ds\le Ct\left(t+\mathbf{E}\int_0^t\|u(s)\|^2ds\right)\end{aligned}$$

hence

$$\|Gu\|^2+\|Ku\|^2=\sup_{t\in I}\mathbf{E}\left(\|Gu(t)\|^2+\|Ku(t)\|^2\right)\le CT^2\left(1+\|u\|^2\right).$$

Remark 2.4 *The linear system (3) is approximate controllable if and only if the operator $\alpha R(\alpha,\Gamma_t^T)$ converges strongly to zero operator as $\alpha\to 0^+$ for any $t\in I$ (see [7],[8]).*

The following lemma gives a formula of a control transferring the state x_0 to an ϵ- neighborhood of an arbitrary state $h\in L_2(\mathfrak{F}_T;X)$.

Lemma 2.5 *For arbitrary $h\in L_2(\mathfrak{F}_T;X)$, $\widetilde{F}(\cdot)\in L_2^{\mathfrak{F}}(I;X)$, and $\widetilde{H}(\cdot)\in L_2^{\mathfrak{F}}(I;L_2^0)$, the control*

$$\begin{aligned}u(t)&=B^*S^*(T-t)R(\alpha,\Gamma_0^T)\left(\mathbf{E}h-S(T)x_0\right)\\ &\quad-B^*S^*(T-t)\int_0^t R(\alpha,\Gamma_s^t)S(T-s)\widetilde{F}(s)ds\\ &\quad-B^*S^*(T-t)\int_0^t R(\alpha,\Gamma_s^t)\Big(S(T-s)\widetilde{H}(s)-\varphi(s)\Big)dw(s)\end{aligned}\tag{5}$$

transfers the system (4) from $x_0\in X$ to

$$\begin{aligned}x(T)&=h-\alpha R(\alpha,\Gamma_0^T)\left(\mathbf{E}h-S(T)x_0\right)+\alpha\int_0^T R(\alpha,\Gamma_s^T)S(T-s)\widetilde{F}(s)ds\\ &\quad+\alpha\int_0^T R(\alpha,\Gamma_s^T)\Big(S(T-s)\widetilde{H}(s)-\varphi(s)\Big)dw(s)\end{aligned}\tag{6}$$

at time T. Here $\varphi(\cdot)\in L_2^{\mathfrak{F}}(I;L_2^0)$ satisfies the Hilbert space version of the representation theorem, that is $h=\mathbf{E}h+\int_0^T\varphi(s)dw(s)$ for any $h\in L_2(\mathfrak{F}_T;X)$.

Proof. See ([10]; Lemma 4). ■

3 Existence and Approximate Controllability Problems

One way to prove the approximate (complete) controllability of nonlinear deterministic or stochastic systems is to show that the nonlinear operator which represents the mild solution of such systems has a fixed point. The first part of this section is to solve the existence problem of the system (1), and then we investigate the approximate controllability of the system. For convenience, let us introduce the notation

$$M_B = \|B\|^2 , \ M_S = \sup_{t\in I} \|S(t)\|^2 .$$

Consider the system

$$\begin{cases} z^\alpha(t) = S(t)x_0 + \int_0^t S(t-s)[Bv^\alpha(s) + F(x,u)(s)]ds + \int_0^t S(t-s)H(x,u)(s)dw(s) \\ v^\alpha(t) = B^*S^*(T-t)R(\alpha,\Gamma_0^T)\left(\mathbf{E}h - S(T)x_0\right) \\ -B^*S^*(T-t)\int_0^t R(\alpha,\Gamma_s^T)S(T-s)F(x,u)(s)ds \\ -B^*S^*(T-t)\int_0^t R(\alpha,\Gamma_s^T)\left[S(T-s)H(x,u)(s) - \varphi(s)\right]dw(s). \end{cases} \tag{7}$$

We use the classical Banach fixed point theorem to show that the nonlinear operator Ψ_α which is given by

$$\Psi_\alpha(x,u)(t) = (z^\alpha(t), v^\alpha(t)) \ t \in I$$

is a contraction mapping on $\mathcal{X} \times \mathcal{U}$ where $(z^\alpha(\cdot), v^\alpha(\cdot))$ as in (7).

Theorem 3.1 *If (A1) and (A2) are satisfied, then the operator $\Psi_\alpha, \alpha > 0$ has a unique fixed point.*

Proof. Firstly, we show that the operator Ψ_α maps $\mathcal{X} \times \mathcal{U}$ into itself. Indeed, if $(x,u) \in \mathcal{X} \times \mathcal{U}$, then by Holder's inequality and Ito's formula we have

$$\mathbf{E}\left\|v^\alpha(t)\right\|^2 \le \frac{5}{\alpha} M_S M_B \Bigg(\|\mathbf{E}h\|^2 + M_S \|x_0\|^2 + T M_S \mathbf{E}\int_0^T \|F(x,u)(s)\|^2 ds$$
$$+ M_S \mathbf{E}\int_0^T \|H(x,u)(s)\|^2 ds + M_S \mathbf{E}\int_0^T \|\varphi(s)\|^2 ds \Bigg)$$

using (A2) and Remark 2.3, the above estimate implies that

$$
\begin{aligned}
&\mathbf{E}\left\|v^{\alpha}(t)\right\|^{2} \\
&\leq \frac{5}{\alpha} M_S M_B \left(\|\mathbf{E}h\|^2 + M_S \|x_0\|^2 + T M_S C \mathbf{E} \int_0^T \left(1 + \|x(s)\|^2 + \|Gu(s)\|^2\right) ds \right. \\
&+ \left. M_S C \mathbf{E} \int_0^T \left(1 + \|x(s)\|^2 + \|Ku(s)\|^2\right) ds + M_S \mathbf{E} \int_0^T \|\varphi(s)\|^2 ds \right) \\
&\leq \frac{5}{\alpha} M_S M_B \left[\|\mathbf{E}h\|^2 + M_S \|x_0\|^2 \right. \\
&+ T^2 M_S C \left(1 + \sup_{t\in I} \mathbf{E}\|x(t)\|^2 + CT^2 \left(1 + \sup_{t\in I} \mathbf{E}\|u(t)\|^2 \right) \right) \\
&+ M_S C T \left(1 + \sup_{t\in I} \mathbf{E}\|x(t)\|^2 + CT^2 \left(1 + \sup_{t\in I} \mathbf{E}\|u(t))\|^2 \right) \right) \\
&\left. + M_S \mathbf{E} \int_0^T \|\varphi(s)\|^2 ds \right]
\end{aligned}
$$

hence there exists a positive constant C_{11} such that

$$
\mathbf{E}\left\|v^{\alpha}(t)\right\|^{2} \leq C_{11} \left(1 + \sup_{t\in I} \mathbf{E}\|x(t)\|^2 + \sup_{t\in I} \mathbf{E}\|u(t)\|^2 \right).
$$

Therefore

$$
\mathbf{E}\left\| \int_0^t S(t-s) B v^{\alpha}(s) ds \right\|^2 \leq M_S M_B C_{11} T^2 \left(1 + \sup_{t\in I} \mathbf{E}\|x(t)\|^2 + \sup_{t\in I} \mathbf{E}\|u(t)\|^2 \right). \tag{8}
$$

Next, applying the same technique, and using (A3), one can show that there exists a positive constant C_{12} such that

$$
\begin{aligned}
&\mathbf{E}\left\| \int_0^t S(t-s) F(x,u)(s) ds \right\|^2 \\
&\leq M_S T \mathbf{E} \int_0^t \|F(x,u)(s)\|^2 ds \\
&\leq M_S C T \mathbf{E} \left(1 + \sup_{t\in I} \mathbf{E}\|x(t)\|^2 + RT^2 \left(1 + \sup_{t\in I} \mathbf{E}\|u(t)\|^2 \right) \right) \\
&\leq C_{12} \left(1 + \sup_{t\in I} \mathbf{E}\|x(t)\|^2 + \sup_{t\in I} \mathbf{E}\|u(t)\|^2 \right).
\end{aligned} \tag{9}
$$

Similarly, there exists a positive constant C_{13} such that

$$\begin{aligned}
&\mathbf{E}\left\|\int_0^t S(t-s)H(x,u)(s)dw(s)\right\|^2 \\
&\leq M_S\mathbf{E}\int_0^t \|H(x,u)(s)\|^2\,ds \\
&\leq M_S C\mathbf{E}\left(1+\sup_{t\in I}\mathbf{E}\,\|x(t)\|^2+RT^2\left(1+\sup_{t\in I}\mathbf{E}\,\|u(t)\|^2\right)\right) \\
&\leq C_{13}\left(1+\sup_{t\in I}\mathbf{E}\,\|x(t)\|^2+\sup_{t\in I}\mathbf{E}\,\|u(t)\|^2\right). \qquad (10)
\end{aligned}$$

Gathering the estimates (8)-(10), we deduce that there exists a positive constant C_1 such that

$$\|\Psi_\alpha(x,u)\|^2=\|v^\alpha\|^2+\|z^\alpha\|^2\leq C_1\left(1+\|x\|^2+\|u\|^2\right).$$

So Ψ_α maps $\mathcal{X}\times\mathcal{U}$ into itself.

Second step is to show that there exists a natural (sufficiently large) natural number n such that Ψ_α^n is a contraction mapping. For this, let (x_1,u_1), and (x_2,u_2) be arbitrary processes in $\mathcal{X}\times\mathcal{U}$. Then

$$\begin{aligned}
&\mathbf{E}\,\|\Psi_\alpha(x_1,u_1)(t)-\Psi_\alpha(x_2,u_2)(t)\|^2 \\
&\leq \mathbf{E}\,\|v_1^\alpha(t)-v_2^\alpha(t)\|^2+\mathbf{E}\,\|z_1^\alpha(t)-z_2^\alpha(t)\|^2 \\
&\leq \mathbf{E}\,\|v_1^\alpha(t)-v_2^\alpha(t)\|^2+3\mathbf{E}\left\|\int_0^t S(t-s)B\left(v_1^\alpha-v_2^\alpha\right)(s)ds\right\|^2 \\
&+3\mathbf{E}\left\|\int_0^t S(t-s)\left(F(x_1,u_1)-F(x_2,u_2)\right)(s)ds\right\|^2 \\
&+3\mathbf{E}\left\|\int_0^t S(t-s)\left(H(x_1,u_1)-H(x_2,u_2)\right)(s)dw(s)\right\|^2 \\
&=I_1(t)+3(I_2(t)+I_3(t)+I_4(t)).
\end{aligned}$$

Each of $I_k, k=1,2,3,$ and 4 can be estimated as following:

$$\begin{aligned}
I_1(t)&=\mathbf{E}\,\|v_1^\alpha(t)-v_2^\alpha(t)\|^2 \\
&\leq 2M_SM_B\mathbf{E}\left\|\int_0^t R(\alpha,\Gamma_s^T)S(T-s)\left(F(x_1,u_1)-F(x_2,u_2)\right)(s)ds\right\|^2 \\
&+2M_SM_B\mathbf{E}\left\|\int_0^t R(\alpha,\Gamma_s^T)S(T-s)\left(H(x_1,u_1)-H(x_2,u_2)\right)(s)dw(s)\right\|^2 \\
&\leq \frac{2}{\alpha}M_S^2M_Bt\mathbf{E}\int_0^t \|\left(F(x_1,u_1)-F(x_2,u_2)\right)(s)\|^2\,ds \\
&+\frac{2}{\alpha}M_S^2M_B\mathbf{E}\int_0^t \|\left(H(x_1,u_1)-H(x_2,u_2)\right)(s)\|^2\,ds
\end{aligned}$$

using (A1) we have

$$
\begin{aligned}
I_1(t) &\leq \frac{2}{\alpha} M_S^2 M_B C t \mathbf{E} \int_0^t \|x_1(s) - x_2(s)\|^2 + \|Gu_1(s) - Gu_2(s)\|^2 \, ds \\
&+ \frac{2}{\alpha} M_S^2 M_B C \mathbf{E} \int_0^t \|x_1(s) - x_2(s)\|^2 + \|Ku_1(s) - Ku_2(s)\|^2 \, ds \\
&= \frac{2}{\alpha} M_S^2 M_B C \, [1+t] \int_0^t \mathbf{E} \, \|x_1(s) - x_2(s)\|^2 \, ds \\
&+ \frac{2}{\alpha} M_S^2 M_B C^2 \, [1+t] \int_0^t s \left(\int_0^s \mathbf{E} \, \|u_1(r) - u_2(r)\|^2 \, dr \right) ds \\
&\leq \frac{2}{\alpha} M_S^2 M_B C t \, [1+t] \, \|x_1 - x_2\|^2 + \frac{2}{\alpha} M_S^2 M_B C^2 t^3 \, [1+t] \, \|u_1 - u_2\|^2 \\
&= \frac{2}{\alpha} M_S^2 M_B C t \, (1+t) \left[\|x_1 - x_2\|^2 + C t^2 \, \|u_1 - u_2\|^2 \right].
\end{aligned}
$$

In similar ways, we have

$$
\begin{aligned}
I_2(t) &= \mathbf{E} \left\| \int_0^t S(t-s) B \left(v_1^\alpha - v_2^\alpha \right)(s) ds \right\|^2 \leq M_S M_B t \mathbf{E} \int_0^t \|v_1^\alpha(s) - v_2^\alpha(s)\|^2 \, ds \\
&\leq \frac{2}{\alpha} M_S^3 M_B^2 C t \int_0^t s \, (1+s) \left[\|x_1 - x_2\|^2 + C s^2 \, \|u_1 - u_2\|^2 \right] ds \\
&\leq \frac{2}{\alpha} M_S^3 M_B^2 C t^3 \, (1+t) \left[\|x_1 - x_2\|^2 + C t^2 \, \|u_1 - u_2\|^2 \right].
\end{aligned}
$$

$$
\begin{aligned}
I_3(t) &= \mathbf{E} \left\| \int_0^t S(t-s) \left(F(x_1, u_1) - F(x_2, u_2) \right)(s) ds \right\|^2 \\
&\leq M_S t \mathbf{E} \int_0^t \|F(x_1, u_1)(s) - F(x_2, u_2)(s)\|^2 \, ds \\
&\leq M_S C t \mathbf{E} \int_0^t \left(\|x_1(s) - x_2(s)\|^2 + \|Gu_1(s) - Gu_2(s)\|^2 \right) ds \\
&\leq M_S C t \left(\mathbf{E} \int_0^t \|x_1(s) - x_2(s)\|^2 \, ds + C \int_0^t s \left(\mathbf{E} \int_0^s \|u_1(r) - u_2(r)\|^2 \, dr \right) ds \right) \\
&\leq M_S C t^2 \left(\|x_1 - x_2\|^2 + C t^2 \, \|u_1 - u_2\|^2 \right)
\end{aligned}
$$

and

$$
\begin{aligned}
I_4(t) &= \mathbf{E}\left\|\int_0^t S(t-s)\left(H(x_1,u_1)-H(x_2,u_2)\right)(s)dw(s)\right\|^2 \\
&\leq M_S\mathbf{E}\int_0^t \left\|\left(H(x_1,u_1)-H(x_2,u_2)\right)(s)\right\|^2 dw(s) \\
&\leq M_S C\mathbf{E}\int_0^t \left(\|x_1(s)-x_2(s)\|^2+\|Ku_1(s)-Ku_2(s)\|^2\right) ds \\
&\leq M_S C\left(\mathbf{E}\int_0^t \|x_1(s)-x_2(s)\|^2 ds + C\int_0^t s\left(\mathbf{E}\int_0^s \|u_1(r)-u_2(r)\|^2 dr\right) ds\right) \\
&\leq M_S Ct\left(\|x_1-x_2\|^2+Ct^2\|u_1-u_2\|^2\right).
\end{aligned}
$$

Summing up the above estimates to get

$$
\begin{aligned}
&\mathbf{E}\left\|\Psi_\alpha(x_1,u_1)(t)-\Psi_\alpha(x_2,u_2)(t)\right\|^2 \\
&\leq \frac{2}{\alpha}M_S^2M_BCt\left(1+t\right)\left[\|x_1-x_2\|^2+Ct^2\|u_1-u_2\|^2\right] \\
&+\frac{6}{\alpha}M_S^3M_B^2Ct^3\left(1+t\right)\left[\|x_1-x_2\|^2+Ct^2\|u_1-u_2\|^2\right] \\
&+3M_SCt^2\left(\|x_1-x_2\|^2+Ct^2\|u_1-u_2\|^2\right) \\
&+3M_SCt\left(\|x_1-x_2\|^2+Ct^2\|u_1-u_2\|^2\right) \\
&=M_SC\left(3+3t+\frac{6}{\alpha}M_S^2M_B^2t^2(1+t)+\frac{2}{\alpha}M_SM_B\left(1+t\right)\right)t\left[\|x_1-x_2\|^2\right. \\
&\left.+Ct^2\|u_1-u_2\|^2\right]
\end{aligned}
$$

hence, if we put
$L(\alpha) = M_SC\left(3+3T+\frac{6}{\alpha}M_S^2M_B^2T^2(1+T)+\frac{2}{\alpha}M_SM_B\left(1+T\right)\right)\max\{1,CT^2\}$
then

$$
\mathbf{E}\left\|\Psi_\alpha(x_1,u_1)(t)-\Psi_\alpha(x_2,u_2)(t)\right\|^2 \leq L(\alpha)t\left(\|x_1-x_2\|^2+\|u_1-u_2\|^2\right)
$$

or on other side it can be rewritten as

$$
\begin{aligned}
&\mathbf{E}\left\|\Psi_\alpha(x_1,u_1)(t)-\Psi_\alpha(x_2,u_2)(t)\right\|^2 \\
&\qquad\qquad \leq L(\alpha)\int_0^t \left(\mathbf{E}\|x_1(s)-x_2(s)\|^2+\mathbf{E}\|u_1(s)-u_2(s)\|^2\right) ds.
\end{aligned}
$$

Thus

$$\begin{aligned}
&\mathbf{E}\left\|\Psi_\alpha^2(x_1,u_1)(t)-\Psi_\alpha^2(x_2,u_2)(t)\right\|^2\\
&=L(\alpha)\int_0^t \mathbf{E}\left\|\Psi_\alpha^2(x_1,u_1)(s)-\Psi_\alpha^2(x_2,u_2)(s)\right\|^2 ds\\
&\le L^2(\alpha)\int_0^t\int_0^s \left(\mathbf{E}\left\|x_1(r)-x_2(r)\right\|^2+\mathbf{E}\left\|u_1(r)-u_2(r)\right\|^2\right) drds\\
&=L^2(\alpha)\frac{t^2}{2!}\left(\|x_1-x_2\|^2+\|u_1-u_2\|^2\right).
\end{aligned}$$

It is obviously now

$$\sup_{t\in I}\mathbf{E}\left\|\Psi_\alpha^n(x_1,u_1)(t)-\Psi_\alpha^n(x_2,u_2)(t)\right\|^2\le L^n(\alpha)\frac{T^n}{n!}\left(\|x_1-x_2\|^2+\|u_1-u_2\|^2\right).$$

For sufficiently large n , $\frac{(L(\alpha)T)^n}{n!}<1$, which implies that Ψ_α^n is a contraction mapping on $\mathcal{X}\times\mathcal{U}$. Then Ψ_α has a unique fixed point $(x^\alpha,u^\alpha)(\cdot)$ in $\mathcal{X}\times\mathcal{U}$ which is the mild solution of (1). ■

Now, we prove the approximate controllability of the system (1). The analogous of the state $x(T)$ (see (6)) in term of the nonlinear system (7) is given by

$$\begin{aligned}
x(T)&=h-\alpha R(\alpha,\Gamma_0^T)\left(\mathbf{E}h-S(T)x_0\right)+\alpha\int_0^T R(\alpha,\Gamma_s^T)S(T-s)F(x,u)(s)ds\\
&+\alpha\int_0^T R(\alpha,\Gamma_s^T)\left(S(T-s)H(x,u)(s)-\varphi(s)\right)dw(s)
\end{aligned}\tag{12}$$

Theorem 3.2 *If (A1)-(A4) are satisfied, then the system (1) is approximate controllable on I.*

Proof. Let $(x^\alpha,u^\alpha)(\cdot)$ in $\mathcal{X}\times\mathcal{U}$ be the solution of the system (7). Then, by using (12), we have

$$\begin{aligned}
\mathbf{E}\left\|x^\alpha(T)-h\right\|^2&\le 4\left\|\alpha R(\alpha,\Gamma_0^T)\right\|^2\left\|\mathbf{E}h-S(T)x_0\right\|^2\\
&+4\mathbf{E}\left\|\int_0^T \alpha R(\alpha,\Gamma_s^T)S(T-s)F(x,u)(s)ds\right\|^2\\
&+4\mathbf{E}\left\|\int_0^T \alpha R(\alpha,\Gamma_s^T)S(T-s)H(x,u)(s)dw(s)\right\|^2\\
&+4\mathbf{E}\left\|\int_0^T \alpha R(\alpha,\Gamma_s^T)\varphi(s)dw(s)\right\|^2\\
&\le 4\left\|\alpha R(\alpha,\Gamma_0^T)\right\|^2\left\|\mathbf{E}h-S(T)x_0\right\|^2+4M_SC\left(T+1\right)\int_0^T\left\|\alpha R(\alpha,\Gamma_s^T)\right\|^2 ds\\
&+4\int_0^T\left\|\alpha R(\alpha,\Gamma_s^T)\right\|^2\mathbf{E}\left\|\varphi(s)\right\|^2 ds.
\end{aligned}$$

Since $\left\|\alpha R(\alpha,\Gamma_s^T)\right\|^2 \leq 1$, and $\left\|\alpha R(\alpha,\Gamma_s^T)\right\|^2 \to 0$ as $\alpha \to 0^+$, for all $0 \leq s < T$, then by the Lebesque dominated convergence theorem, we have $\mathbf{E}\left\|x^\alpha(T)-h\right\|^2 \to 0$, as $\alpha \to 0^+$ which implies the result. ■

4 Complete Controllability

In this section we study the complete controllability for the stochastic integrodifferential system (1) by assuming (A1), (A2), (A3) and (A4') where (A4') is stated as following

(A4') The linear operator $L_0^T : L_2^{\mathfrak{F}}(I;U) \to L_2(\mathfrak{F}_T;X)$ defined by

$$L_0^T u = \int_0^T S(T-s)Bu(s)ds$$

induces a boundedly invertible operator $\widetilde{L}$ defined on $L_2^{\mathfrak{F}}(I;U)/\ker L_0^T$.

Again, the Banach fixed point theorem can be used to show that the operator

$$\Psi(x,u)(t) = (z(t),v(t)) \ t \in I$$

is a contraction mapping on $\mathcal{X}\times\mathcal{U}$ where $(z(\cdot),v(\cdot))$ is given by

$$\begin{cases} z(t) = S(t)x_0 + \int_0^t S(t-s)[Bv(s)+F(x,u)(s)]ds + \int_0^t S(t-s)H(x,u)(s)dw(s) \\ v(t) = \mathbf{E}\left\{\widetilde{L}^{-1}\left(h - S(T)x_0 - \int_0^T S(T-s)F(x,u)(s)ds - \int_0^T S(T-s)H(x,u)(s)dw(s)\right)|\mathfrak{F}_t\right\}. \end{cases} \tag{13}$$

Theorem 4.1 *If (A1)-(A3) and (A4') are satisfied. Then the system (13) is completely controllable on I.*

Proof. It can be shown that there exist positive constants L_1 and L_2 such that

$$\begin{cases} \mathbf{E}\left\|v(t)\right\|^2 \leq L_1\left(1+\|x\|^2+\|u\|^2\right) \\ \mathbf{E}\left\|v_1(t)-v_2(t)\right\|^2 \leq L_2 t\left[\|x_1-x_2\|^2+\|u_1-u_2\|^2\right]. \end{cases}$$

The rest of the whole proof is similar to that one of Theorem 3.1, hence it is omitted. Therefore Ψ is a contraction mapping on $\mathcal{X}\times\mathcal{U}$ and then it has a unique fixed point $(x,u)(\cdot)$ in $\mathcal{X}\times\mathcal{U}$ which is the solution of (13). Thus the system (1) is complete controllable on I. ■

Example 4.2 *Consider the following stochastic integrodifferential system*

$$\left\{\begin{array}{l} dx_t(t,\theta) = [x_{\theta\theta}(t,\theta) + Bu(t,\theta) + f(t, x(t,\theta), \int\limits_0^t g(s, u(s,\theta))ds)]dt + dw(t) \\ x(t,0) = x(t,\pi) = 0,\ t \in I, 0 < \theta < \pi. \end{array}\right\} \tag{14}$$

Let $X = L_2[0,\pi]$, and let $A : X \to X$ be operator defined by

$$Az = z''$$

with domain

$$D(A) = \{z \in X \mid z, z' \text{are absolutely continuous}, z'' \in X, z(0) = z(\pi) = 0\}.$$

Then

$$Az = \sum_{n=1}^{\infty}(-n^2)\langle z, e_n\rangle e_n(\theta), \qquad z \in D(A),$$

where $e_n(\theta) = \sqrt{2/\pi}\sin(n\theta)$, $0 \le \theta \le \pi$, $n = 1, 2, 3, ...$It is known that A generates an analytic semigroup $S(t), t > 0$ in X and it is given by

$$S(t)z = \sum_{n=1}^{\infty} e^{-n^2 t}\langle z, e_n\rangle e_n(\theta), \qquad z \in X.$$

Now define an infinite-dimensional space

$$U = \left\{ u = \sum_{n=2}^{\infty} u_n e_n(\theta) \middle| \sum_{n=2}^{\infty} u_n^2 < \infty \right\}$$

with a norm defined by

$$\|u\|^2 = \sum_{n=2}^{\infty} u_n^2$$

and a linear continuous mapping B from U to X as follows:

$$Bu = 2u_2 e_1(\theta) + \sum_{n=2}^{\infty} u_n e_n(\theta).$$

It is obvious that $u(t,\theta,\omega) = \sum_{n=2}^{\infty} u_n(t,\omega)e_n(\theta)$ and $Bu(t,\theta,\omega) = 2u_2(t,\omega)e_1(\theta) + \sum_{n=2}^{\infty} u_n(t,\omega)e_n(\theta)$ are in $L_2^{\mathfrak{F}}(I;U)$. Moreover, if $v = \sum_{n=1}^{\infty} v_n e_n(\theta)$ and $x = \sum_{n=1}^{\infty} x_n e_n(\theta)$,

then

$$\begin{cases} B^*v = (2v_1 + v_2)e_2(\theta) + \sum_{n=3}^{\infty} v_n e_n(\theta) \\ B^*S^*(t)x = \left(2x_1 e^{-t} + x_2 e^{-4t}\right) e_2(\theta) + \sum_{n=3}^{\infty} x_n e^{-n^2 t} e_n(\theta). \end{cases}$$

Hence, if $\|B^*S^*(t)x\| = 0$, then $\|2x_1e^{-t} + x_2e^{-4t}\|^2 + \sum_{n=3}^{\infty} \left\|x_n e^{-n^2 t}\right\|^2 = 0$ which implies $x_n = 0$ for all $n \geq 1$, i.e. $x = 0$. By ([4], Theorem 4.17) the deterministic linear system corresponding to the system (14) is approximately controllable on I, which implies that the corresponding linear stochastic system is approximately controllable on I. Remark 2.4 implies that the operator $\alpha R(\alpha, \Gamma_t^T) \to 0$ as $\alpha \to 0^+$. Hence by Theorem 3.2, the system (14) is approximate controllable on I, provided that f and g satisfy the hypotheses (A1)-(A3).

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