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The Relation Among Bishop Spherical Indicatrix Curves

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Abstract

In this paper, we study on spherical images of Bishop vector fields of a regular curve which lies on the unit sphere in 3-dimensional Euclidean space. The Bertrand mates and spherical involte curves of each spherical image curve are found. Also the spherical indicatrix curve of Bishop Darboux vector which is called Bishop Darboux spherical indicatrix is defined.

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1 Introduction

Many researchers use the Frenet frame of a curve to characterize the properties of curves. This thought aroused that there can be other frames which have same advanteges with the Frenet frame. Then, in 1975, R.L. Bishop introduced the Bishop frame or parallel transport frame which can be uncovered by tangent vector and two parallel vectors of the curve. After the invention of the Bishop frame, many studies were performed in [3], [5], [6], [8], [9], [10].

In [4] spherical images of the vectors of Frenet frame in Euclidean space are founded and also the cases of spherical image curves relative to each other in n-dimensional Euclidean space were investigated in [1]. The new spherical images of a regular curve which are called Bishop spherical images are determined by using Bishop frame vectors in [8].

We define another Bishop spherical image which is called *Bishop Darboux spherical indicatrix* and investigate these four spherical images in the light of spherical involute curves and Bertrand mates.

2 Preliminaries

E^3 is Euclidean 3-space which is provided with the standart metric given by

$$g(x,x) = \langle x,x\rangle = x_1^2 + x_2^2 + x_3^2$$

where $x = (x_1, x_2, x_3)$ is a vector in E^3. The norm of a vector v is given by $\|v\| = \sqrt{g(v,v)}$ and for the velocity vector v of a curve if $\|v\| = 1$, then the curve is called *unit speed curve*. Two vectors v and w are called *orthogonal* if and only if $g(v,w) = 0$. Let $\alpha(s)$ be a curve in E^3. If $\|\alpha'(s)\| = 1$, then s is *arc-length parameter* of the curve where $\alpha'(s)$ is the tangent vector of $\alpha(s)$. Let $\{T, N, B\}$ be the moving Frenet frame of a unit speed curve $\alpha(s)$ in E^3. T, N, B are called respectively, *tangent vector*, *principal normal vector* and *binormal vector*. These vectors are expressed by

$$\begin{aligned} T &= \alpha'(s) \\ N &= \frac{\alpha''(s)}{\|\alpha''(s)\|} \\ B &= T \times N \end{aligned}$$

where $T \times N$ is the vectoral product of T and N (see[7]). The function $\kappa = \|\alpha''(s)\|$ is called the *curvature* and $\tau = \langle N', B\rangle$ is called the *torsion* of α. Additionally, the Frenet equations are given in [7] as in the matrix form

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

The sphere with origin center and radius $r\rangle 0$ in E^3 is defined as

$$S^2 = \left\{ x = (x_1, x_2, x_3) \in E^3 \mid \langle x,x\rangle = r^2 \right\}.$$

The spherical indicatrix curves are formed by translating Frenet vectors of a regular curve to the center of unit sphere which is S^2 with $r = 1$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. $\{T, N_1, N_2\}$ Bishop frame is constructed by taking tangent vector T of a regular curve which is unique and choosing any arbitrary basis $\{N_1, N_2\}$. This basis is in the normal plane which is perpendicular to T and the derivatives of N_1 and N_2 are depend on only T ,not each other [6]. Also the Bishop formulas are given in [2], [3] as

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

where k_1 and k_2 are *Bishop curvatures* ; $\kappa(s) = \sqrt{k_1^2 + k_2^2}$, $\theta(s) = \arctan \frac{k_2}{k_1}$ and $\tau(s) = -\frac{d\theta(s)}{ds}$. The relation between Frenet and Bishop frames can be written as

$$\begin{aligned} T &= T \\ N &= \cos\theta(s) N_1 + \sin\theta(s) N_2 \\ B &= -\sin\theta(s) N_1 + \cos\theta(s) N_2 \end{aligned}$$

and Bishop curvatures are

$$\begin{aligned} k_1 &= \kappa \cos\theta(s) \\ k_2 &= \kappa \sin\theta(s) \end{aligned}$$

[8]. Let two regular curves be α and β in E^3. $\{V_1, V_2, V_3\}$ and $\{V_1^*, V_2^*, V_3^*\}$ are Frenet frames of α and β , respectively. If the tangent vectors of these curves are perpendicular to each other, ie $\langle V_1, V_1^* \rangle = 0$, then the curve α is *involute* of β. If the principal normal vectors are linearly dependent, ie $V_2 = \lambda V_2^*$; $\lambda \in R$, then α and β are called *Bertrand mates.*

3 Bishop Darboux Spherical Indicatrix Curve and Results

Let α be a unit speed curve with arc lenght parameter s and $\{V_1, V_2, V_3\}$ be the Frenet frame of α. The spherical images of Frenet vectors will be shown as α_i ;

$$\begin{aligned} \alpha_i : I &\longrightarrow S^2 \\ s &\longrightarrow \alpha_i(s) = V_i(s); \quad 1 \leq i \leq 3 \end{aligned}$$

which are called Frenet spherical indicatrix curves. As a notation, we call $\{T, N_1, N_2\}$ Bishop frame of α, also k_1 and k_2 are Bishop curvatures of α.

We now notice the Bishop Darboux vector of α and its spherical indicatrix curve.

As it is indicated in [3], if a rigid body moves along a regular curve, then the motion of body consists of translation and rotation along the curve α. The rotation is determined by an angular velocity vector ω which is called the *Bishop Darboux vector* and given by

$$\omega = -k_2 N_1 + k_1 N_2.$$

ω satisfies these equations;

$$\begin{aligned} T' &= \omega \times T \\ N_1' &= \omega \times N_1 \\ N_2' &= \omega \times N_2. \end{aligned}$$

If we unitize the Bishop Darboux vector, we get

$$C = \frac{\omega}{\|\omega\|} = \frac{-k_2 N_1 + k_1 N_2}{\sqrt{k_1^2 + k_2^2}}.$$

We obtain a spherical indicatrix curve which is called *Bishop Darboux spherical indicatrix of the curve* α by translating the unit vector C to the center of unit sphere S^2.

Let $\alpha_C(s) = C(s)$, with the arc lenght parameter s_C , be Bishop Darboux spherical indicatrix of the curve $\alpha(s)$. By differentiating α_C, we get

$$\begin{aligned} \frac{d\alpha_C}{ds} &= \frac{dC}{ds} \\ \alpha_C^{\bullet} \frac{ds_c}{ds} &= C' \end{aligned}$$

and the derivative of C can be calculated as

$$C' = \frac{k_2 k_1 k_1' - k_2' k_1^2}{(k_1^2 + k_2^2)^{\frac{3}{2}}} N_1 + \frac{k_1' k_2^2 - k_1 k_2 k_2'}{(k_1^2 + k_2^2)^{\frac{3}{2}}} N_2.$$

We have the tangent vector of Bishop Darboux spherical indicatrix and denote it as T_{α_C}

$$\begin{aligned} T_{\alpha_C} &= \frac{k_2 k_1 k_1' - k_2' k_1^2}{\sqrt{(k_2 k_1 k_1' - k_2' k_1^2)^2 + (k_1' k_2^2 - k_1 k_2 k_2')^2}} N_1 \\ &\quad + \frac{k_1' k_2^2 - k_1 k_2 k_2'}{\sqrt{(k_2 k_1 k_1' - k_2' k_1^2)^2 + (k_1' k_2^2 - k_1 k_2 k_2')^2}} N_2 \end{aligned}$$

where

$$\frac{ds_c}{ds} = \sqrt{\frac{(k_2k_1k_1' - k_2'k_1^2)^2 + (k_1'k_2^2 - k_1k_2k_2')^2}{(k_1^2+k_2^2)^3}}.$$

Now we can give the theorems which express the relation of spherical indicatrix curves.

Theorem 3.1 *Let α be a regular curve in 3-dimensional Euclidean space. Both of N_1 and N_2 spherical indicatrix curves of α are spherical involutes for tangent and Bishop Darboux spherical indicatrices of α.*

Proof. Let us denote the tangent vectors of tangent, N_1 and N_2 spherical indicatrices as T_ξ , T_δ and T_ψ respectively. These tangent vectors are given in [8] by

$$\begin{aligned} T_\xi &= \frac{k_1N_1 + k_2N_2}{\sqrt{k_1^2+k_2^2}} \\ T_\delta &= T \\ T_\psi &= T \end{aligned}$$

where $\{T, N_1, N_2\}$ is Bishop frame and k_1 and k_2 are Bishop curvatures of α.

If the inner products are calculated, we have

$$\begin{aligned} \langle T_\xi, T_\delta\rangle &= 0 \\ \langle T_{\alpha_C}, T_\delta\rangle &= 0 \end{aligned}$$

and

$$\begin{aligned} \langle T_\xi, T_\psi\rangle &= 0 \\ \langle T_{\alpha_C}, T_\psi\rangle &= 0. \end{aligned}$$

The tangent vectors of N_1 and N_2 spherical indicatrices are perpendicular to tangent vectors of tangent and Bishop Darboux spherical indicatrices. So the proof is completed.

Theorem 3.2 *Let α be a regular curve in 3-dimensional Euclidean space. N_1 and N_2 spherical indicatrix curves of α are Bertrand mates.*

Proof. Let us denote the principal normal vectors of tangent, N_1 and N_2 spherical indicatrices as N_ξ , N_δ and N_ψ respectively.

The principal normal vectors are given in [8] by

$$\begin{aligned} N_\xi &= \frac{1}{\kappa_\xi}\left\{-T+\frac{k_2^3}{(k_1^2+k_2^2)^2}\left(\frac{k_1}{k_2}\right)' N_1+\frac{k_1^3}{(k_1^2+k_2^2)^2}\left(\frac{k_2}{k_1}\right)' N_2\right\} \\ N_\delta &= -\frac{N_1}{\kappa_\delta}-\frac{k_2}{k_1\kappa_\delta}N_2 \\ N_\psi &= -\frac{k_1}{k_2\kappa_\psi}N_1-\frac{N_2}{\kappa_\psi} \end{aligned}$$

where

$$\begin{aligned} \kappa_\xi &= \sqrt{1+\left[\frac{k_2^3}{(k_1^2+k_2^2)^2}\left(\frac{k_1}{k_2}\right)'\right]^2+\left[\frac{k_1^3}{(k_1^2+k_2^2)^2}\left(\frac{k_2}{k_1}\right)'\right]^2} \\ \kappa_\delta &= \sqrt{1+\left(\frac{k_2}{k_1}\right)^2} \\ \kappa_\psi &= \sqrt{1+\left(\frac{k_1}{k_2}\right)^2}. \end{aligned}$$

By putting curvatures κ_δ and κ_ψ of N_1 and N_2 spherical indicatrices, we have the principal normal vectors as

$$\begin{aligned} N_\delta &= -\frac{k_1}{\sqrt{k_1^2+k_2^2}}N_1-\frac{k_2}{\sqrt{k_1^2+k_2^2}}N_2 \\ N_\psi &= -\frac{k_1}{\sqrt{k_1^2+k_2^2}}N_1-\frac{k_2}{\sqrt{k_1^2+k_2^2}}N_2 \end{aligned}$$

It can be seen that $N_\delta = N_\psi$, so the principal normal vectors of N_1 and N_2 spherical indicatrices are linearly dependent. As a result of this, they are Bertrand mates.

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