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On the Construction of New Iterative Methods with Fourth-Order Convergence by Combining Previous Methods

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Abstract

In this paper, we present some new modification of Newton's method for solving nonlinear equations. Analysis of convergence shows that these methods have order of convergence four. Several numerical examples are given to illustrate the performance of the presented methods.

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1 Introduction

One of the most important problems in numerical analysis is solving nonlinear equations. To solve these equations, we can use iterative methods such as Newton's method and its variants. In this paper, we consider iterative methods to find a simple root of a non-linear equation $f(x) =$, where $f : I \to \mathbf{R}$ for an open interval I is a scalar function.
The classical Newton's method for a single non-linear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

This is a powerful and well-known iterative method known to convergence quadratically. Recently, there has been some progress on iterative methods with higher-order of convergence that they require the computation of lower-order derivatives, [2,4,5,6,7,8,10,11].
One of the third-order modifications of Newton's method, [5], is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n)}{f'(x_n)}, \tag{2}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3}$$

also, a third-order variant of Newton's method, [4], is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(y_n)}{f'(x_n) + f'(y_n)}. \tag{4}$$

Another third-order method, [4], that is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)f(y_n)}{[f(x_n) - f(y_n)]f'(x_n)}. \tag{5}$$

In this paper, we give two new iterative methods with fourth-order convergence. The rest of this paper is organized as follows. It has been shown that some of the existing third-order iteration formulas may be used to construct new fourth-order methods that this is the main purpose of this paper. In section 2, we give new iterative methods. In section 3, we compare the results with some numerical examples and in the last section, the conclusion is presented.

2 The methods and analysis of convergence

In this section, for construction of new iterative methods, we use iterative methods given by Eqs. (2)-(5). The following theorems show the order of convergence of these methods i.e., iterative methods given by Eqs. (2)-(3), Eq. (4) and Eq. (5).

Theorem 2.1 *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to \mathbf{R}$ for an open interval I. Then, the new method that is defined by Eqs. (2)-(3) has the third-order convergence and satisfies the following error equation*

$$e_{n+1} = 2c_2^2 e_n^3 + o(e_n^4) \tag{6}$$

where $e_n = x_n - \alpha$ and $c_2 = \frac{f''(\alpha)}{2f'(\alpha)}$.

Theorem 2.2 *[4]. Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to R$ for an open interval I. Then, the method that is defined by Eq. (4) has the third-order convergence and it then satisfies the order equation*

$$e_{n+1} = 2c_2^3 e_n^3 + o(e_n^4) \tag{7}$$

where $e_n = x_n - \alpha$ and $c_k = \frac{f^{(k)}(\alpha)}{f'(\alpha)k!}$.

Theorem 2.3 *[4]. Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to \mathbf{R}$ for an open interval I. Then, the new method that is defined by Eq. (5) has the third-order convergence and satisfies the following error equation,*

$$e_{n+1} = c_2^2 e_n^3 + o(e_n^4), \tag{8}$$

where $e_n = x_n - \alpha$ and $c_2 = \frac{f''(\alpha)}{2f'(\alpha)}$.

Now, we construct a new iterative method by combining Eqs. (2) and (4).

$$x_{n+1} = A[x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n)}{f'(x_n)}] + B[x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(y_n)}{f'(x_n) + f'(y_n)}]$$

where $A = -1$, $B = 2$.
It is clear that we can write above equation as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + f(y_n)(\frac{1}{f'(x_n)} - \frac{4}{f'(x_n) + f'(y_n)}). \tag{9}$$

Now we show that the order of convergence of iterative method that is defined by Eq. (9) is at least four.

Theorem 2.4 *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to R$ for an open interval I. Then, the new method that is defined by Eq. (9) has the fourth-order convergence and satisfies the following error equation,*

$$e_{n+1} = 3c_2^3 e_n^4 + o(e_n^5) \tag{10}$$

where $e_n = x_n - \alpha$ and $c_k = \frac{f^{(k)}(\alpha)}{f'(\alpha)k!}$.

Proof. Let α be a simple zero of f. By the Taylor expansions,

$$f(x_{n)} = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + o(e_n^5)], \tag{11}$$

and

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + o(e_n^5)], \tag{12}$$

where $c_k = \frac{f^{(k)(\alpha)}}{f'(\alpha)k!}$, $k = 2, 3, ...$and $e_n = x_n - \alpha$. Dividing Eq. (11) by Eq. (12) gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 3c_4 - 4c_2^3 e_n^4 + o(e_n^5), \tag{13}$$

and hence,

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 3c_4 + 4c_2^3)e_n^4 + o(e_n^5). \tag{14}$$

Again, expanding $f(y_n)$ and $f'(y_n)$ about α and then using Eq. (14), we have

$$f(y_n) = f'(\alpha)[c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 3c_4 + 5c_2^3)e_n^4 + o(e_n^5)], \tag{15}$$

and

$$f'(y_n) = f'(\alpha)[1 + 2c_2 e_n^2 + 4c_2(c_3 - c_2^2)e_n^3 + c_2(-11c_2c_3 + 6c_4 + 8c_2^3)e_n^4 + o(e_n^5)]. \tag{16}$$

After an elementary calculation, we obtain

$$\frac{f(y_n)}{f'(x_n)} = c_2 e_n^2 + 2(-2c_2^2 + c_3)e_n^3 + (-14c_2c_3 + 13c_2^3 + 3c_4)e_n^4 + o(e_n^5), \tag{17}$$

$$\begin{aligned} f'(x_n) + f'(y_n) = f'(\alpha)[2 + 2c_2 e_n + (2c_2^2 + 3c_3)e_n^2 + 4c_4 e_n^3 + (8c_2^4 - 11c_2^2 c_3 \\ +6c_2c_4 + 5c_5)e_n^4) + o(e_n^5)], \end{aligned} \tag{18}$$

$$\frac{f(y_n)}{f'(x_n) + f'(y_n)} = \frac{1}{2}[c_2 e_n^2 + (-3c_2^2 + 2c_3)e_n^3 + (7c_2^3 - \frac{21}{2}c_2c_3 + 3c_4)e_n^4] + o(e_n^5). \tag{19}$$

From Eqs. (12), (17), (18), and (19), we obtain

$$\frac{f(x_n)}{f'(x_n)} - f(y_n)(\frac{1}{f'(x_n)} - \frac{4}{f'(x_n) + f'(y_n)}) = -3c_2^3 e_n^4 + o(e_n^5). \tag{20}$$

Thus,

$$\begin{aligned} x_{n+1} = x_n - [\frac{f(x_n)}{f'(x_n)} - f(y_n)(\frac{1}{f'(x_n)} - \frac{4}{f'(x_n) + f'(y_n)})], \\ e_{n+1} + \alpha = e_n + \alpha - [e_n - 3c_2^3 e_n^4 + o(e_n^5)], \\ e_{n+1} = 3c_2^3 e_n^4 + o(e_n^5). \end{aligned} \tag{21}$$

Equation (21) establishes the fourth-order convergence of method that is defined by Eq. (9).

For construction of second iterative method, we use iterative methods, defined in Eqs. (2) and (5). If we combine these methods, we have

$$x_{n+1} = A[x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n)}{f'(x_n)}] + B[x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)f(y_n)}{[f(x_n) - f(y_n)]f'(x_n)}]$$

where $A = -1$, $B = 2$.
Now we can write above equation as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{f(y_n)}{f'(x_n)} - 2\frac{f(x_n)f(y_n)}{f'(x_n)[f(x_n) - f(y_n)]}. \tag{22}$$

Theorem 2.5 *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to R$ for an open interval I. Then, the new method that is defined by Eq. (22) has the fourth-order convergence and satisfies the following error equation,*

$$e_{n+1} = (-13c_2c_3 + 5c_2^3 + 6c_4)e_n^4 + o(e_n^5), \tag{23}$$

where $e_n = x_n - \alpha$ and $c_k = \frac{f^{(k)}(\alpha)}{f'(\alpha)k!}$.

Proof. Let α be a simple zero of f. By the Taylor expansions,

$$f(x_n) - f(y_n) = f'(\alpha)[e_n + (-c_3 + 2c_2^2)e_n^3 + (-5c_2^3 + 7c_2c_3 - 2c_4)e_n^4 + o(e_n^5)], \tag{24}$$

and

$$\frac{f(y_n)}{f(x_n) - f(y_n)} = c_2e_n + 2(c_3 - c_2^2)e_n^2 + (3c_2^3 - 6c_2c_3 + 3c_4)e_n^3 + (9c_2^4 - 13c_2^2c_3 + 2c_2c_4 + 2c_3^2)e_n^4 + o(e_n^5), \tag{25}$$

From Eqs. (13), (17), and (25), we obtain

$$\frac{f(x_n)}{f'(x_n)} - \frac{f(y_n)}{f'(x_n)} + 2\frac{f(x_n)f(y_n)}{f'(x_n)[f(x_n) - f(y_n)]} = -(-13c_2c_3 + 5c_2^3 + 6c_4)e_n^4 + o(e_n^5), \tag{26}$$

Thus,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{f(y_n)}{f'(x_n)} - 2\frac{f(x_n)f(y_n)}{f'(x_n)[f(x_n) - f(y_n)]},$$

$$e_{n+1} + \alpha = e_n + \alpha - [e_n - (-13c_2c_3 + 5c_2^3 + 6c_4)e_n^4 + o(e_n^5),$$

$$e_{n+1} = (-13c_2c_3 + 5c_2^3 + 6c_4)e_n^4 + o(e_n^5). \tag{27}$$

Equation (27) establishes the fourth-order convergence of method which is defined by Eq. (22) .

Clearly, per iteration of Eqs. (9) and (22) require two function's evaluations and two first derivative's evaluations. If we consider the definition of efficiency index, [9] , as $p^{\frac{1}{m}}$, where p is the order of the method and m is the number of function's evaluations in per iteration, we can have a method that is defined by the Eq. (9) that it has the efficiency index equals to $4^{\frac{1}{4}} \approx 1.414$ and a method that is defined by the Eq. (22) has the efficiency index equals to $4^{\frac{1}{3}} \approx 1.587$, which is better than the Newton's method with efficiency index equals to $\sqrt{2} \approx 1.414$.

3 Numerical Examples

In this section, all computations were done using **MATHEMATICA** by using 64 digit floating point arithmetic (Digits:=64). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs:

$$(i)|x_{n+1} - x_n| < \epsilon, \qquad (ii)|f(x_{n+1})| < \epsilon,$$

and so, when the stopping criterion is satisfied, x_{n+1} is taken as an approximate of the exact root α. For numerical illustrations, we used the fixed stopping criterion $\epsilon = 10^{-15}$.

We present some numerical test results to illustrate the efficiency of the new iterative method in Table 1.

We compare the Newton's method, (NM), Changbum Chun's method with $\beta = \frac{1}{2}$, (CM1), [8],which is defined by

$$x_{n+1} = y_n - \frac{f^2(x_n)}{f^2(x_n) - 2f(x_n)f(y_n) + 2\beta f^2(y_n)} \frac{f(y_n)}{f'(x_n)}, \tag{28}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{29}$$

where $\beta \in \mathbf{R}$, Changbum Chun's method with $\beta = 1$, (CM2), [8],

$$x_{n+1} = y_n - \frac{f^3(x_n)}{f^3(x_n) - 2f^2(x_n)f(y_n) + 2\beta f(x_n)f^2(y_n) - 2\beta^2 f^3(y_n)} \frac{f(y_n)}{f'(x_n)}, \tag{30}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{31}$$

where $\beta \in \mathbf{R}$.

Changbum Chun's method, (CM3), [5], that is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{[f'(x_n)^2]}, \tag{32}$$

Kou's method [2] , (KM1),

$$x_{n+1} = x_n - \frac{f^2(x_n) + f^2(y_n)}{f'(x_n)(f(x_n) - f(y_n))}, \tag{33}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{34}$$

and so, King's method with $\beta = 3$, (KM2), [7],

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)}, \tag{35}$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{36}$$

and our methods that is defined by Eq. (9), (ESM1) and Eq. (22), (ESM2), that introduced in the present contribution. We used the following test functions:

$$f_1(x) = x^3 + 4x^2 - 10,$$

$$f_2(x) = \sin^2 x - x^2 + 1,$$

$$f_3(x) = x^2 - e^x - 3x + 2,$$

$$f_4(x) = \cos x - x,$$

$$f_5(x) = (x-1)^3 - 1,$$

$$f_6(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,$$

$$f_7(x) = \sin x - x/2,$$

$$f_8(x) = (x^3 + 4x^2 - 10)^2,$$

Table 1. Comparison of the number of iterations (NIT) in (NM), (CM1), (CM2), (CM3) , (KM1) , (KM2) , (ESM1), and (ESM2) methods.

	NIT							
$f(x)$	NM	CM1	CM2	CM3	KM1	KM2	ESM1	ESM2
$f_1(x), x_0 = 1.5$	5	3	3	3	3	3	3	3
$f_2(x), x_0 = 2.$	6	4	4	4	4	4	4	4
$f_3(x), x_0 = 2$	6	4	4	4	4	4	4	4
$f_4(x), x_0 = 1.7$	5	4	4	4	4	4	4	4
$f_5(x), x_0 = 3.5$	8	5	5	5	5	5	5	5
$f_6(x), x_0 = -2$	9	5	5	6	6	6	6	6
$f_7(x), x_0 = 2.3$	6	4	4	4	4	4	4	4
$f_8(x), x_0 = 1.4$	46	26	26	28	27	29	26	27

4 Conclusion

In this paper, we presented some new modification of Newton's method. In Theorems 4 and 5, we proved that the order of convergence of the methods are four and analysis of efficiency shown that these methods could compete with Newton's method.

References

[1] E. Babolian and J. Biazzar, Solution of nonlinear equations by modified Adomian decomposition method, *Appl. Math. Comput,* 132 (2002), 162 -172.

[2] C. Chun, A family of composite fourth-order iterative methods for solving non-linear equations, *Appl. Math. Comput,* 187 (2007), 951- 956.

[3] C. Chun, A simply constructed third-order modification of Newton's method, *J. Comput. Appl. Math,* 219 (2008), 81- 89.

[4] C. Chun, Construction of Newton-like iteration methods for solving nonlinear equations, *Numer. Math,* 104 (2006), 297-315.

[5] C. Chun, Iterative method improving Newton's method by the decomposition method, *Computers Math. Applic,* 50 (2005), 1559- 1568.

[6] C. Chun, Some fourth-order iterative mthods for solving non-linear equations, *Appl. Math. Comput,* 195 (2008), 454-459.

[7] C. Chun, Yoon Mee Ham, Some fourth-order modification of Newton's method, *Appl. Math. Comput,* 197 (2008), 654-658.

[8] C. Chun, Some variants of King's fourth-order family of methods for nonlinear equations, *Appl. Math. Comput,* 190 (2007), 57-62.

[9] W. Gautschi, *Numerical Analysis:An introduction,* Birkhäser, 1997.

[10] Li Tai-fang, Li De-sheng, Xu Zhao-di and Fang Ya-ling, New iterative methods for non-linear, *Appl. Math. Comput,* 197 (2008), 755-759.

[11] X. Wu and D. Fu, New high-order convergence iteration methods without employing derivatives for solving non-linear equation, *Computers Math. Applic,* 41 (3/4) (2001), 489-495.

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