A Note on Weakly S-Supplemented Subgroups of Finite Groups

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Abstract

A subgroup of a finite group $G$ is called S-permutable in $G$ if it permutes with every Sylow subgroup of $G$. A subgroup $H$ of a finite group $G$ is called weakly S-supplemented in $G$ if $G$ has a subgroup $T$ such that $HT = G$, $H \cap T \leq H_{sG}$, where $H_{sG}$ is the largest S-permutable subgroup of $G$ contained in $H$. In this paper, we investigate further the influence of weakly S-supplemented of certain subgroups of prime power order on the structure of finite groups. Some recent results are improved and generalized.

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1 Introduction

Throughout only finite groups are considered. The terminology and notions employed agree with standard usage, as in Doerk and Hawkes in [5].

Recall that a subgroup $H$ of a group $G$ is called S-permutable, S-quasinormal or $\pi$-quasinormal in $G$ if $HP = PH$ for all Sylow subgroups $P$ of $G$. This concept was introduced by Kegel in [9] and has been studied extensively by Deskins in [4]. Following Wang [13], a subgroup $H$ of a group $G$ is called c-normal in $G$ if $G$ has a normal subgroup $T$ such that $HT = G$ and $T \cap H \leq H_G$ where $H_G$ is
As a generalization of c-normality, Ballester-Bolinshes et. al. [2] introduced the following definition: A subgroup $H$ of a group $G$ is called c-supplemented in $G$ if $G$ has a subgroup $T$ such that $HT = G$ and $T \cap H \leq H_G$. In [12], Skiba introduced the following definitions: 1) A subgroup $H$ of a group $G$ is called weakly $S$-permutable in $G$ if $G$ has a subnormal subgroup $T$ such that $HT = G$ and $T \cap H \leq H_{sG}$; 2) A subgroup $H$ of a group $G$ is called weakly $S$-supplemented in $G$ if $G$ has a subgroup $T$ such that $HT = G$ and $T \cap H \leq H_{sG}$. It is clear that all the subgroups, whether they are normal subgroups, c-normal subgroups, c-supplemented subgroups, $S$-permutable subgroups or weakly $S$-permutable subgroups, are weakly $S$-supplemented.

Of late there has been a considerable interest to investigate the structure of a finite group $G$ under the assumption that certain subgroups are well-situated in the group. For example, Buckely [3] proved that if $G$ is a group of odd order and all minimal subgroups of $G$ are normal in $G$, then $G$ is supersolvable (recall that a subgroup of prime order is called a minimal subgroup). Later on Shaalan [11] proved that if $G$ is a finite group and every subgroup of prime order or of order 4 is $S$-permutable in $G$, then $G$ is supersolvable. Afterward, Li [10] proved that if a finite group $G$ possesses a normal subgroup $N$ of odd order such that $G/N$ is supersolvable, and if, for each Sylow subgroup $P$ of $N$, every minimal subgroup of $P$ is normal in $N_G(P)$, then $G$ is supersolvable. Next, Wang [13] proved that a finite group $G$ is supersolvable if every subgroup of $G$ of prime order or of order 4 is c-normal in $G$. Recently, Wenbin et. al. [14] have shown that a finite group $G$ is supersolvable if every subgroup of any noncyclic Sylow subgroup of $G$ of prime order or of order 4 is weakly $S$-supplemented (weakly $S$-permutable) in $G$. A related results are also proved in Ezzat Mohamed [6] and in Huang and Li [7].

The purpose of this paper is to go further in the investigation of the influence of weakly $S$-supplemented of certain subgroups of prime power order on the structure of groups. Most of the above mentioned results are improved and generalized.

2 Preliminaries

In this section, we collect some results that will be used later.

Lemma 2.1 Let $G$ be a group and $H \leq K \leq G$. Then:
(i) If $H$ is weakly $S$-supplemented in $G$, then $H$ is weakly $S$-supplemented in $K$.

(ii) Suppose that $H$ is normal in $G$. Then $K/H$ is weakly $S$-supplemented in $G/H$ if and only if $K$ is weakly $S$-supplemented in $G$.

(iii) Suppose that $H$ is normal in $G$. Then the subgroup $HE/H$ is weakly $S$-supplemented in $G/H$ for every weakly $S$-supplemented subgroup $E$ in $G$ satisfying $(|H|,|E|)=1$.

**Proof.** See [12, Lemma 2.10]. □

**Lemma 2.2** Let $G$ be a group which is not $p$-nilpotent but all its proper subgroups are $p$-nilpotent. Then $G$ is a minimal non-nilpotent group ($G$ is not nilpotent but all its proper subgroups are nilpotent).

**Proof.** See [8, IV, Satz 5.4]. □

**Lemma 2.3** Let $G$ be a minimal non-nilpotent group. Then:

(i) $|G|=p^nq^m$, $p$ and $q$ are distinct primes; the Sylow $p$-subgroup is normal in $G$; the Sylow $q$-subgroups are non-normal cyclic and for every Sylow $q$-subgroup $Q$ of $G$, $\Phi(Q) \leq Z(G)$.

(ii) The class of the Sylow $p$-subgroup $P$ is at most two; $\Phi(P) \leq Z(G)$.

(iii) For $p > 2$, $P$ has exponent $p$ and, for $p = 2$, the exponent of $P$ is at most 4.

**Proof.** See [8, III, Satz 5.2]. □

**Lemma 2.4** Let $p$ be the smallest prime divisor of the order of a group $G$. If every subgroup of $G$ of order $p$ or 4 (if $p = 2$) is weakly $S$-supplemented in $G$, then $G$ is $p$-nilpotent.

**Proof.** See [6, Theorem 3.1]. □

If $P$ is a finite $p$-group, we denote

$$\Omega(P) = \Omega_1(P) \text{ if } p > 2 \quad \text{and} \quad \Omega(P) = \langle \Omega_1(P), \Omega_2(P) \rangle \text{ if } p = 2,$$

where

$$\Omega_i(P) = \langle x \in P : O(x) = p^i \rangle.$$

**Lemma 2.5** Let $P$ be a Sylow 2-subgroup of a finite group $G$. If $P$ is quaternion-free and $\Omega_1(P) \leq Z(G)$, then $G$ is 2-nilpotent.
Proof. See [1, Corollary 2]. □

Lemma 2.6 Let $G$ be a group and let $K \triangleleft G$ such that $G/K$ is supersolvable. If all subgroups of $K$ of prime order or of order 4 are weakly S-supplemented in $G$, then $G$ is supersolvable.

Proof. See [6, Theorem 3.2]. □

3 Main Results

We define $D(G) = \cap\{H : H \triangleleft G \text{ and } G/H \text{ is nilpotent}\}$ and call it the nilpotent residual of $G$.

Theorem 3.1 Let $p$ be the smallest prime dividing the order of $G$ and let $P$ be a Sylow $p$-subgroup of $G$. If $P$ is quaternion-free and all minimal subgroups of $D(G) \cap P$ are weakly S-supplemented in $G$, then $G$ is $p$-nilpotent.

Proof. Assume that the result is false and let $G$ be a counterexample of minimal order. Then $G$ is not $p$-nilpotent. Knowing that all its Sylow $p$-subgroups are conjugate in $G$, we deduce that the hypothesis of our theorem is subgroup-closure by Lemma 2.1. Consequently, $G$ contains a minimal non $p$-nilpotent subgroup $K$, say, (that is, every proper subgroup of $K$ is $p$-nilpotent but $K$ itself is not $p$-nilpotent). Now, by Lemma 2.2, $K$ is a minimal non-nilpotent group. By Lemma 2.3, $|K| = p^n q^m$ where $q$ is a prime different from $p$, $K$ has a normal Sylow $p$-subgroup $K_p$ of exponent $p$ when $p$ is odd and of exponent at most 4 when $p = 2$ and a non-normal cyclic Sylow $q$-subgroup $K_q$. Without loss of generality, we can assume that $K_p \leq P$. Clearly, $D(K) = K_p$. Then $D(K) \cap K_p = K_p$. By hypothesis, all minimal subgroups of $K_p$ are weakly S-supplemented in $G$. Then, by Lemma 2.1, all minimal subgroups of $K_p$ are weakly S-supplemented in $K$. In fact, if the exponent of $K_p$ is $p$, then $K$ is $p$-nilpotent by Lemma 2.4; a contradiction. Thus, the exponent of $P$ is 4 and hence $p = 2$. So, by Lemma 2.3, $K'_2 = Z(K_2) = \Phi(K_2)$, $K_2$ is elementary abelian and $K_2/K'_2$ is a chief factor of $K$. Then $\Omega_1(K_2) = K_2' \leq Z(K_2)$. Now, by applying Lemma 2.5, we conclude that $K$ is 2-nilpotent; a final contradiction. □

The following example shows that the hypothesis ”$P$ is quaternion-free” is necessary in Theorem 3.1:

Example 3.2 Set $G = SL(2,3)$. Then the Sylow 2-subgroup $P$ of $G$ is the quaternion group of order 8 and $D(G) = P$. Thus all minimal subgroups
of \(D(G) \cap P\) are normal (weakly \(S\)-supplemented) in \(G\). However, \(G\) is not 2-nilpotent.

As immediate consequences of Theorem 3.1, we have:

**Corollary 3.3** Let \(p\) be the smallest prime dividing the order of \(G\) and let \(P\) be a Sylow \(p\)-subgroup of \(G\). If all subgroups of \(D(G) \cap P\) of order \(p\) or 4 (if \(p = 2\)) are weakly \(S\)-supplemented in \(G\), then \(G\) is \(p\)-nilpotent.

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**Corollary 3.5** Let \(p\) be the smallest prime dividing the order of \(G\) and let \(P\) be a Sylow \(p\)-subgroup of \(G\). If all subgroups of \(D(G) \cap P\) of order \(p\) or 4 (if \(p = 2\)) are \(c\)-normal in \(G\), then \(G\) is \(p\)-nilpotent.

**Corollary 3.6** Let \(p\) be the smallest prime dividing the order of \(G\) and let \(P\) be a Sylow \(p\)-subgroup of \(G\). If all subgroups of \(D(G) \cap P\) of order \(p\) or 4 (if \(p = 2\)) are \(c\)-supplemented in \(G\), then \(G\) is \(p\)-nilpotent.

**Corollary 3.7** Let \(p\) be the smallest prime dividing the order of \(G\) and let \(P\) be a Sylow \(p\)-subgroup of \(G\). If all subgroups of \(D(G) \cap P\) of order \(p\) or 4 (if \(p = 2\)) are \(S\)-permutable in \(G\), then \(G\) is \(p\)-nilpotent.

**Corollary 3.8** Let \(p\) be the smallest prime dividing the order of \(G\) and let \(P\) be a Sylow \(p\)-subgroup of \(G\). If all subgroups of \(D(G) \cap P\) of order \(p\) or 4 (if \(p = 2\)) are weakly \(S\)-permutable in \(G\), then \(G\) is \(p\)-nilpotent.

Now we can prove:

**Theorem 3.9** If all minimal subgroups of \(D(G) \cap P\) are weakly \(S\)-supplemented in \(G\) for all Sylow subgroups \(P\) of \(G\), then \(G\) is supersolvable or \(G\) has a section isomorphic to the quaternion group of order 8.

**Proof.** If \(G\) has a section isomorphic to the quaternion group of order 8, then we are done. Thus we can assume that \(G\) has no section isomorphic to the quaternion group of order 8. Theorem 3.1 implies that \(G\) is \(r\)-nilpotent, where \(r\) is the smallest prime dividing the order of \(G\). Then \(G = RK\) where \(R\) is a Sylow \(r\)-subgroup of \(G\) and \(K\) is a normal Hall \(r'\)-subgroup of \(G\). Clearly, \(D(K) \leq D(G)\). Since all minimal subgroups of \(D(G) \cap P\) are weakly \(S\)-supplemented
in $G$, we have, by hypothesis and Lemma 2.1, that all minimal subgroups of $D(K') \cap K_P$ are weakly S-supplemented in $K$ for all Sylow subgroups $K_P$ of $K$. By induction on the order of $G$, $K$ is supersolvable. Hence $Q$ is a normal Sylow $q$-subgroup of $K$, where $q$ is the largest prime dividing the order of $K$. Since $Q \lhd K \lhd G$, we have that $Q \lhd G$.

Now, we show that $G/Q$ satisfies the hypothesis of the theorem. Put $D(G/Q) = L/Q$. Since $(G/Q)/(L/Q) \cong G/L$ is nilpotent, we have that $D(G) \leq L$ and since

$$(G/Q)/(D(G)Q/Q) \cong G/D(G)Q \cong (G/D(G))/(D(G)Q/Q)$$

is nilpotent, we have that $L \leq D(G)Q$. Hence $L = D(G)Q$ and so $D(G/Q) = D(G)Q/Q$. Clearly,

$$D(G/Q) \cap (PQ/Q) = (D(G)Q/Q) \cap (PQ/Q) = (D(G) \cap P)Q/Q),$$

for all Sylow subgroups $PQ/Q$ of $G/Q$. By hypothesis and Lemma 2.1, all minimal subgroups of $D(G/Q) \cap (PQ/Q)$ are weakly S-supplemented in $G/Q$ for all Sylow subgroups $PQ/Q$ of $G/Q$. Thus $G/Q$ satisfies the hypothesis of the theorem.

By induction on the order of $G$, $G/Q$ is supersolvable. Since $G/D(G)$ is nilpotent (in particular, supersolvable), we have that $G/(D(G) \cap Q)$ is supersolvable. Applying Lemma 2.6 yields $G$ is supersolvable which completes the proof of the theorem. □

The argument which established Theorem 3.9 can be easily adapted to obtain the following corollaries:

**Corollary 3.10** If all subgroups of $D(G) \cap P$ of prime order or of order 4 are weakly S-supplemented in $G$ for all Sylow subgroups $P$ of $G$, then $G$ is supersolvable.

**Corollary 3.11** [14]. Let $G$ be a group. If every subgroup of any non-cyclic Sylow subgroup of $G$ of prime order or of order 4 is weakly S-supplemented in $G$, then $G$ is supersolvable.

**Corollary 3.12** [3]. Assume that $G$ is a group of odd order and every subgroup of $G$ of prime order is normal in $G$, then $G$ is supersolvable.

**Corollary 3.13** [11]. If all subgroups of $P$ of prime order or of order 4 are S-permutable in $G$ for all Sylow subgroups $P$ of $G$, then $G$ is supersolvable.
Corollary 3.14 [13]. If every subgroup of a group \( G \) of prime order or of order 4 is c-normal in \( G \), then \( G \) is supersolvable.

Corollary 3.15 [2]. If every subgroup of a group \( G \) of prime order or of order 4 is c-supplemented in \( G \), then \( G \) is supersolvable.

Corollary 3.16 [12]. If every subgroup of a group \( G \) of prime order or of order 4 is weakly \( S \)-permutable in \( G \), then \( G \) is supersolvable.

References


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