International Mathematical Forum, Vol. 6, 2011, no. 27, 1313 – 1318

# Solitary and Traveling Wave Solutions to a Model of Long Range Diffusion Involving Flux with Stability Analysis

**Manar A. Al-Qudah**

Math Department, Rabigh Faculty of Science & Art
King Abdul Aziz University, P. O. Box 344
Rabigh 2191, Saudi Arabia
manarqudah@ahoo.com

**Marwan S. Abualrub**

Math Department ,The University of Jordan
P. O. Box 11942, Amman-Jordan
abualrubms@yahoo.com

**Abstract**

A model for insect dispersal has been considered, an equilibrium and stability analysis has been done and the behavior to the solitary and traveling wave solutions of the model are obtained.

**Mathematics Subject Classification:** 92B99

**Keywords:** Solitary wave, traveling wave, stability analysis

## 1. Introduction

The dynamics of population has been described using mathematical models which have been very successful in giving good effect in the study of animal and human populations. Fife [5], considered reaction and diffusion systems which are distributed in 3-dimentional spaces. Abualrub [1], studied diffusion in two dimensional spaces for which diffusion is more realistic and applicable in life.
Also he talked about long range diffusion with population pressure in Plankton-

Herbivore populations. In [3], we included long range diffusion involving flux for insect population and talked about the existence and uniqueness of solutions for the considered model in the $L^{p,q}$ space. And we found the required p and q in similar approach used in [2]. In this paper we study solitary wave solution using the generalized Tanh function method as in [7]. Also we find a traveling wave solution then, we discuss stability of solutions to our model.

## 2. Long Range Diffusion Involving Flux

Here we consider long range diffusion involving flux in two dimensions which is given by:

$$u_t - D\Delta^{(2)}u = \alpha_1 u + \alpha_2 u^2 + \alpha_3 u_x + \alpha_4 \Delta\left(u^{\alpha+1}\right) \qquad (1)$$

$$u(x,0) = f(x) \qquad (2)$$

; where $u = u(x,t)$ is the insect population density $x \in R^2$. Here $\Delta$ represents the Laplacian operator and

$$\Delta^{(2)} = \sum_{i,j}^{2} \frac{\partial^4}{\partial x_i^2 \partial x_j^2}. \qquad (3)$$

$u_t$ is the rate of change of the insect population density, $D\Delta^{(2)}u$ is the long range diffusion term, where $D$ is a small constant, and $\alpha$, $\alpha_4$ are positive constants. $u^2$ is the interaction between the males and females of the insect population, and $u_x$ is the instantaneous flux in the $x$ direction due to molecular diffusion. Here $\Delta\left(u^{\alpha+1}\right)$ is the regular diffusion of the insect population.

For simplicity take the one dimensional case thus, equations (1) and (2) become

$$u_t - Du_{xxxx} = \alpha_1 u + \alpha_2 u^2 + \alpha_3 u_x + 2\alpha_4\left(u_x^2 + uu_{xx}\right) \qquad (4)$$

$$u(x,0) = f(x); \qquad \text{where } x \in R \qquad (5)$$

## 3. Solitary Wave Solutions

First, we want to find an exact solitary wave solution to equation (4) using the generalized Tanh function method which is based on the **Riccati** equation which is given by

$$y' = A + By^2 \qquad (6)$$

; where $y' = \dfrac{dy}{dz}$, and $A$, $B$ are constants.

The main idea of this method is mentioned in [7].

Letting

$$u(x,t) = u(\xi); \quad \text{where} \quad \xi = x - ct \qquad (7)$$

substituting equation (7) into equation (4) we obtain

$$(\alpha_3 + c)u' + Du^{(4)} + \alpha_1 u + \alpha_2 u^2 + 2\alpha_4 u'^2 + 2\alpha_4 uu'' = 0 \qquad (8)$$

which is a fourth order nonlinear ODE.
Now, by introducing the independent variable $Y = \tanh\xi$ the solution of equation (8) can be written in the following form

$$u(x,t) = u(\xi) = \sum_{j=0}^{n} a_j Y^j \tag{9}$$

; where $Y = \tanh\xi$ ;$n$ and $a_0, a_1, \ldots, a_n$ can be determined as in the description of the tanh method which is mentioned earlier.
In equation (9) balancing the term $uu''$ with the term $u^{(4)}$ gives $n = 2$; that is the solution has the form

$$u(\xi) = a_0 + a_1 Y + a_2 Y^2 . \tag{10}$$

Substituting (6) and (10) into (9) to get the following difficult algebraic equation

$$\begin{aligned}
&(\alpha_3 + c)\left[a_1 A + a_1 BY^2 + 2a_2 AY + 2a_2 BY^3\right] \\
&+ D\begin{bmatrix} 16a_2 A^3 B + 16a_1 A^2 B^2 Y + 136a_2 A^2 B^2 Y^2 + 40a_1 AB^3 Y^3 + 240a_2 AB^3 Y^4 + 24a_1 B^4 Y^5 \\ +120a_2 B^4 Y^6 \end{bmatrix} \\
&+ \alpha_1\left[a_0 + a_1 Y + a_2 Y^2\right] + \alpha_2\left[a_0 + a_1 Y + a_2 Y^2\right]^2 + 2\alpha_4\left[a_1 A + a_1 BY^2 + 2a_2 AY + 2a_2 BY^3\right]^2 \\
&+ 2\alpha_4\left[a_0 + a_1 Y + a_2 Y^2\right]\left[2a_2 A^2 + 2a_1 ABY + 8a_2 ABY^2 + 2a_1 B^2 Y^3 + 6a_2 B^2 Y^4\right] \\
&= 0
\end{aligned} \tag{11}$$

Now, collecting the coefficients to get a system of seven nonlinear algebraic equations. Solving the resulting system for $a_0, a_1, a_2, c$ using mathematica software, we obtain the following set of solutions

$$a_0 = \frac{D\alpha_2}{2\alpha_4^2} - \frac{4DAB}{\alpha_4}$$

$$a_1 = 0$$

$$a_2 = \frac{-6DB^2}{\alpha_4}$$

$$c = -\alpha_3$$

We can choose one of the set of solutions for the Riccati equation, see[4]
; namely the following:

$$\begin{aligned}
u(\xi) &= a_0 + a_2\left(\tanh\xi + i\operatorname{sech}\xi\right)^2 \\
&= \left(\frac{D\alpha_2}{2\alpha_4^2} - \frac{D}{\alpha_4}\right) - \frac{3D}{2\alpha_4}\left(\tanh\xi + i\operatorname{sech}\xi\right)^2
\end{aligned} \tag{12}$$

**Conclusion 3.1**

If we assume that $D = 1$, $\alpha_2 = 1$, $\alpha_4 = 1$, then the solution can be graphed using mathematica software to be an ellipse; this means that our solution is stable since the origin is a center point, see [4].

## 4. Traveling Wave Solution

Now want to seek an exact traveling wave solution. If such solution exists it can be written in the following form:

$$u(x,t) = k(z), \quad z = x - ct \tag{13}$$

;where $c$ is the wave speed. Substituting equation (13) in equation ( 4) to obtain:

$$(\alpha_3 + c)k' + Dk^{(4)} + \alpha_1 k + \alpha_2 k^2 + 2\alpha_4\left(k'^2 + kk''\right) = 0 \tag{14}$$

; where the differentiation in equation (14) is with respect to $z$.
Since we are looking for a traveling wave solution, we have to impose the following conditions on $k$:

$$k(-\infty) = 1 \quad \text{and} \quad k(\infty) = 0 \tag{15}$$

**Remark 4.1**
The reason for imposing the boundary conditions (15) is because we seek a nonnegative solution $k$ of equation (14), for which $k$ at one end, say as $z \to -\infty$, is at one steady state and as $z \to +\infty$ it is at the other. As done in Murray [6] and from the first term in the asymptotic wave front solution to Fisher's equation we expect that the solution of equation (14) together with the conditions in (15) might take the form:

$$k(z) = \frac{1}{\left(1 + ae^{z}\right)^4} \tag{16}$$

; where, as in Murray [6], we must assume that $k(0) = \frac{1}{2}$ and this will give $a = \sqrt[4]{2} - 1$.
**Remark 4.2**
The conditions on $k$ given in equation (16) and the condition $k(0) = \frac{1}{2}$ means that the number of insects in the beginning of the experiment was one unit that is larger than its number in the middle of the experiment because half of the insects died out and at the end of the experiment they will become all dead.

## 5. Stability of Solutions

Here we study the stability of solutions to a model of long range diffusion involving flux. Consider the equation (14) which can be convert into a system of ODE's as follows:

$$\begin{aligned}
k_1' &= k_2 \\
k_2' &= k_3 \\
k_3' &= k_4 \\
k_4' &= -\frac{1}{D}\left[(\alpha_3 + c)k_2 + 2\alpha_4\left[k_2^2 + k_1 k_3\right] + \alpha_2 k_1^2 + \alpha_1 k_1\right]
\end{aligned} \tag{17}$$

Then system (17) can be written in the matrix form as follows:

$$\begin{bmatrix} k_1' \\ k_2' \\ k_3' \\ k_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\alpha_1}{D} & \frac{-(\alpha_3+c)}{D} & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{2\alpha_4}{D}\left[k_2^2 + k_1k_3\right] + \alpha_2 k_1^2 \end{bmatrix} \quad (18)$$

equation (18) can be written in the following form

$$\vec{k}' = A\vec{k} + g(\vec{k}) \quad (19)$$

the last system is almost linear system since $g(\vec{x})$ has continuous ($1^{st}$,$2^{nd}$, $3^{rd}$, and $4^{th}$) partial derivatives, and (0, 0, 0, 0) is the only equilibrium point of the system. Now, we need to study the type of the equilibrium point (0, 0, 0, 0) and discuss its stability, to do this we find

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ \frac{-\alpha_1}{D} & \frac{-(\alpha_3+c)}{D} & 0 & -\lambda \end{vmatrix},$$

to get the following equation:

$$\lambda^4 + \frac{(\alpha_3 + c)}{D}\lambda + \frac{\alpha_1}{D} = 0. \quad (20)$$

Solving equation (20) using mathematica software to get very complicated roots, see [4] .

**Remark 5.1**

We can use our result in finding the solitary wave solution by generalized tanh function method $c = -\alpha_3$. Then, equation (20) can be written as:

$$\lambda^4 + \frac{\alpha_1}{D} = 0 \quad (21)$$

Solving equation (21) and assuming that both $\alpha_1$ and D are positive, we obtain the following roots:

$$\lambda_1 = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{\alpha_1}{D}\right)^{1/4}$$

$$\lambda_2 = \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{\alpha_1}{D}\right)^{1/4}$$

$$\lambda_3 = \left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(\frac{\alpha_1}{D}\right)^{1/4}$$

$$\lambda_4 = \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(\frac{\alpha_1}{D}\right)^{1/4}$$

**Conclusion 5.2**

We may assume that $D = 1$, as in the solitary wave solution. It is obvious that the equilibrium point $(0,0,0,0)$ will be unstable spiral point because $\lambda_1$ and $\lambda_4$ have positive real parts.

## References

[1] M. S. Abualrub, Long Range Diffusion-Reaction Model on Population Dynamics, Documenta Mathematica, 3(1998), 331-340.

[2] M. S. Abualrub, Existence and Uniqueness of Solutions to a Diffusive Predator-Prey Model, Journal of Applied Functional Analysis, 4(1). Copyright (2009) Eudoxus press, LLC, 107-111.

[3] M. A. Al- Qudah, and M. S. Abualrub, Existence of Solutions to a Model of Long Range Diffusion Involving Flux, Journal of Applied Functional Analysis. , 5(4). Copyright (2010) Eudoxus press, LLC, 370-376.

[4] M. A. Al- Qudah, Stability of Solution for Non-linear Partial Differential Equations, Ph. d thesis, The University of Jordan, Amman, Jun. 2010.

[5] P. C. Fife, Stationary Patterns for Reaction-Diffusion Equations, Pitman Research Notes in Mathematics, eds., W. E. Fitzgibbon, III and H. W. Walker. 14(1977), 81-121.

[6] J. D. Murray, Mathematical Biology, Biomathematics Texts, Springer-Verlage , New York, 1989.

[7] L. Wazzan, More Soliton Solutions to the KDV and the KDV-Burgers Equations Proc. Pakistan Acad. Sci. 44(2)(2007), 117-120.

Received: October, 2010