On Simple Singular AP-Injective Modules

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Abstract

In this paper, we investigate some properties of rings whose simple (singular) right $R$–modules are AP-injective. It is proved that an MERT ring $R$ is von Neumann regular if and only if $R$ is a right weakly continuous ring whose simple singular right $R$–modules are AP-injective. It is also proved that a right quasi-duo ring $R$ is von Neumann regular if and only if every right $R$–module is AP-injective if and only if every cyclic right $R$–module is AP-injective if and only if every simple right $R$–module is AP-injective if and only if $R$ is a strongly regular ring. It is shown that if $R$ is a ring whose every simple singular right $R$–module is AP-injective, then $J(R)$ contains no nonzero nilpotent elements if and only if $J(R) = 0$.

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1 Introduction

Throughout the paper, $R$ is an associative ring with identity and $M$ is a right $R$-module with $S = \text{End}(M_R)$. For $a \in R$, $r(a)$, $l(a)$ denote the right annihilator and left annihilator of $a$, respectively. We write $J(R)$, $Z_r(R)(Z_l(R))$ for the Jacobson radical, the right(left) singular ideal. $X \leq M$ denote that $X$ is a submodule of $M$.

Recall a module $M_R$ with $S = \text{End}(M_R)$ is said to be almost principally injective (or AP-injective for short), if for any $a \in R$, there exists a left $S$–submodule $X_a$ of $M_R$ such that $l_M r_R(a) = Ma \oplus X_a$. AP-injectivity has been generally studied(see [1,2,3]). Actually, many authors investigated some properties of rings whose every simple right $R$–module(resp. every simple singular right $R$–module) is GP-injective[3-9]. Since a right AP-injective ring need not be right GP-injective, in this paper, we study some properties on the rings whose every simple(or simple singular) right $R$–module is AP-injective.
2 Main Results

These are the main results of the paper.

Lemma 2.1 Suppose $M$ is a right $R$–module with $S = \text{End}(M_R)$. If $l_Mr_R(a) = Ma \oplus X_a$, where $X_a$ is a left $S$–submodule of $M_R$. Set $f : aR \to M$ is a right $R$-homomorphism, then $f(a) = ma + x$ with $m \in M$, $x \in X_a$.

Proof. Since $f(a)r_R(a) = f(ar_R(a)) = f(0) = 0$, so $r_R(a) \subseteq r_R(f(a))$, thus $l_Mr_R(f(a)) \subseteq l_Mr_R(a) = Ma \oplus X_a$, and $f(a) \in l_Mr_R(f(a))$, hence $f(a) = ma + x$ with $m \in M_R$, $x \in X_a$.

Recall that a ring $R$ is called MERT(or MELT), if every essential right(or left) ideal of $R$ is a two-sided ideal. A ring $R$ is called right weakly continuous if $J(R) = Z_r(R)$, $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$. A ring $R$ is called right(left) weakly regular if $I^2 = I$ for each right(left) ideal $I$ of $R$, equivalently $a \in aRaR(a \in RaRa)$ for every $a \in R$. $R$ is weakly regular if it is both right and left weakly regular. A ring $R$ is called right quasi-duo [10] if every maximal right ideal of $R$ is a two-sided ideal.

Theorem 2.2 For an MERT ring $R$, the following statements are equivalent.

(1) $R$ is von Neumann regular.

(2) $R$ is a right weakly continuous ring whose simple singular right $R$–modules are AP-injective.

Proof. (1)$\Rightarrow$(2) Observe that if $R$ is von Neumann regular, then every right $R$–module is AP-injective. So we are done.

(2)$\Rightarrow$(1) Suppose that $Z_r(R) \neq 0$. Then by [1, Lemma 1], we may assume that $Z_r(R)$ is not reduced. So there exists nonzero $a \in Z_r(R)$ such that $a^2 = 0$. We claim that $Z_r(R) + r(a) = R$. If not, there exists a maximal essential right ideal $M$ containing $Z_r(R) + r(a)$. Thus $R/M$ is AP-injective, then $l_{R/M}r_{R}(a) = (R/M)a \oplus X_a, X_a \leq R/M$. Let $f : aR \to R/M$ be defined by $f(ar) = r + M$. By Lemma 2.1, $1 + M = f(a) = ca + M + x, c \in R, x \in X_a, 1 - ca + M = x \in R/M \cap X_a = 0$, so $1 - ca \in M$. Since $R$ is an MERT ring, $ca \in M$. Hence $1 \in M$, which is a contradiction. Therefore $Z_r(R) + r(a) = R$.

Thus we can write $1 = c + d$ for some $c \in Z_r(R)$ and $d \in r(a)$. Thus $a = ca$ and so $(1 - c)a = 0$. Since $c \in Z_r(R) = J(R)$, $1 - c$ is invertible. Thus $a = 0$, which is a contradiction. Therefore $Z_r(R)$ is reduced and so $Z_r(R) = 0$.

Theorem 2.3 If every simple right $R$–module is AP-injective, then $R$ is a right weakly regular ring.
Proof. We will show that \( RaR + r(a) = R \) for any \( a \in R \). Suppose that there exists \( b \in R \) such that \( RbR + r(b) \neq R \), then there exists a maximal right ideal \( M \) of \( R \) containing \( RbR + r(b) \). Thus \( R/M \) is AP-injective, then \( l_{R/M} r_{R}(b) = (R/M)b \oplus X_b, X_b \leq R/M \). Let \( f : bR \to R/M \) be defined by \( f(br) = r + M \). Note \( f \) is well-defined. So \( 1 + M = f(b) = cb + M + x, c \in R, x \in X_b, 1 - cb + M = x \in R/M \cap X_b = 0, 1 - cb \in M, cb \in M \), and hence \( 1 \in M \), which is a contradiction. Therefore \( RaR + r(a) = R \) for any \( a \in R \), then \( R \) is right weakly regular.

Recall that \( R \) is a ZI ring if for \( a, b \in R, ab = 0 \) implies \( aRb = 0 \). A ring \( R \) is called idempotent reflexive if \( aRe = 0 \) implies \( eRa = 0 \) for \( a, e = e^2 \in R \)(see [4]).

**Corollary 2.4** Let \( R \) be a ZI ring. If every simple right \( R \)-module is AP-injective, then \( R \) is a weakly regular ring.

**Proof.** By [6, Lemma 3] and Theorem 2.3, it is easy.

**Theorem 2.5** Let \( R \) be an idempotent reflexive ring. If every simple singular right \( R \)-module is AP-injective, then \( R \) is right weakly regular.

**Proof.** We need only to prove \( RaR + r(a) = R \) for any \( a \in R \). If not, then there exists a maximal right ideal \( M \) of \( R \) containing \( RaR + r(a) \). By [4, Lemma 6], \( M \) is essential. So \( R/M \) is a simple singular right \( R \)-module, \( R/M \) is right AP-injective. Then by the same method as in the proof of Theorem 2.3, \( R \) is right weakly regular.

**Theorem 2.6** If \( R \) is right quasi-duo, then the following statements are equivalent:

1. Every right \( R \)-module is AP-injective.
2. Every cyclic right \( R \)-module is AP-injective.
3. Every simple right \( R \)-module is AP-injective.
4. \( R \) is a strongly regular ring.
5. \( R \) is a von Neumann regular ring.
6. \( R \) is a right weakly regular ring.

**Proof.** Obviously (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) and (4) \( \Rightarrow \) (5), (5) \( \Rightarrow \) (1). (4) \( \Leftrightarrow \) (6) is clear by [11, Proposition 4.7]. Thus it remains to prove that (3) implies (4). For any \( 0 \neq a \in R \), we will show \( aR + r(a) = R \). Suppose not. Then there exists a maximal right ideal \( M \) of \( R \) containing \( aR + r(a) \). Since \( R/M \) is simple, \( l_{R/M} r_{R}(a) = (R/M)a \oplus X_a, X_a \leq R/M \). Let \( f : aR \to R/M \) be defined by \( f(ar) = 1 + r \). Note that \( f \) is well-defined. Thus there exists \( c \in R, x \in X_a \) such that \( 1 + K = f(a) = ca + M + x, 1 - ca + M = x \in R/M \cap X_a = 0, 1 - ca \in M \). Since \( R \) is right quasi-duo, \( ca \in M \), thus \( 1 \in M \), which is a contradiction. Therefore \( aR + r(a) = R \). So \( R \) is strongly regular.
Corollary 2.7  Let \( R \) be a right quasi-duo ring. Then the following statements are equivalent.

1. \( R \) is von Neumann regular.
2. Every simple right \( R \)-module is injective.
3. Every simple right \( R \)-module is \( P \)-injective.
4. Every simple right \( R \)-module is \( GP \)-injective.
5. Every simple right \( R \)-module is \( AP \)-injective.

Proof. By Theorem 2.6 and [5, Corollary 11], it is clear.

Theorem 2.8  If \( R \) is a ZI ring, then the following statements are equivalent:

1. \( R \) is a strongly regular ring.
2. \( R \) is an MELT ring whose every simple left \( R \)-module is \( AP \)-injective.
3. \( R \) is an MERT ring whose every simple right \( R \)-module is \( AP \)-injective.
4. \( R \) is an MELT ring whose every simple singular left \( R \)-module is \( AP \)-injective.
5. \( R \) is an MERT ring whose every simple singular right \( R \)-module is \( AP \)-injective.

Proof. Obviously (1)\( \Rightarrow \) (3)\( \Rightarrow \) (5) and (1)\( \Rightarrow \) (2)\( \Rightarrow \) (4). (4)\( \Rightarrow \) (1) and (5)\( \Rightarrow \) (1) is similar. We need only to prove (4)\( \Rightarrow \) (1). Suppose (4), for any \( 0 \neq a \in R \), we will show that \( Ra + l(a) = R \). If not, then there exists a maximal left idea \( M \) of \( R \) containing \( Ra + l(a) \), and \( M \) is an essential left ideal of \( R \), thus \( R/M \) is left \( AP \)-injective, then as the proof in Theorem 2.6, \( R \) is strongly regular.

A ring \( R \) is called weakly right duo [13] if for any \( a \in R \), there exists a positive integer \( n \) such that \( a^nR \) is a two-sided ideal. In [14], Yu proved that a weakly right(left) duo ring was right(left)quasi-duo.

Theorem 2.9  The following statements are equivalent:

1. \( R \) is strongly regular.
2. \( R \) is a weakly right duo ring whose simple singular right \( R \)-modules are \( AP \)-injective.
3. \( R \) is an abelian right quasi-duo ring whose simple singular right \( R \)-modules are \( AP \)-injective.

Proof. (1)\( \Rightarrow \) (2) is clear. (2)\( \Rightarrow \) (1)Since \( R \) is a weakly right duo ring, \( R \) is a right quasi-duo ring. It is clear by Theorem 2.6.

(1)\( \Rightarrow \) (3) is clear. (3)\( \Rightarrow \) (1)\( R \) is a right quasi-duo ring, and \( R \) is abelian by [13, Lemma 4]. Thus \( R \) is right weakly regular by Theorem 2.12. Hence \( R \) is a strongly regular ring by [11, Proposition 4.7].

Theorem 2.10  If \( R \) is an abelian right quasi-duo( in particular, weakly right duo) ring, then the following statements are equivalent.

1. Every singular right \( R \)-module is \( AP \)-injective.
(2) Every cyclic singular right $R$-module is AP-injective.
(3) Every simple singular right $R$-module is AP-injective.
(4) $R$ is a strongly regular ring.
(5) $R$ is a von Neumann regular ring.
(6) $R$ is a right weakly regular ring.

Proof. The trivial implications are $(1) \Rightarrow (2) \Rightarrow (3)$ and $(4) \Rightarrow (5) \Rightarrow (6)$. By Theorem 2.9, $(3) \Rightarrow (4)$. $(6) \Rightarrow (4)$ is clear by [11, Proposition 4.7]. If $R$ is von Neumann regular, then every right $R$-module is P-injective, so every right $R$-module is AP-injective, thus $(5) \Rightarrow (1)$. This completes the proof.

Corollary 2.11 If $R$ is an abelian right quasi-duo (in particular, weakly right duo) ring, then the following statements are equivalent.

(1) $R$ is von Neumann regular.
(2) Every simple singular right $R$-module is injective.
(3) Every simple singular right $R$-module is P-injective.
(4) Every simple singular right $R$-module is GP-injective.
(5) Every simple singular right $R$-module is AP-injective.

Proof. By Theorem 2.10 and [6, Corollary 11], it is easy to be proved.

Lemma 2.12 If $R$ is a ring whose every simple singular right $R$-module is AP-injective, then $J(R) \cap Z(R)$ contains no nonzero nilpotent elements.

Proof. Take $b \in J(R) \cap Z(R)$ with $b^2 = 0$. If $b \neq 0$, then $r(b) + RbR$ is an essential right ideal of $R$. We will prove that $r(b) + RbR = R$. If not, there exists a maximal essential right ideal $M$ of $R$ containing $r(b) + RbR$. By assumption, $R/M$ is right AP-injective, thus $l_{R/M}r_{R}(b) = (R/M)b + X_b$, $X_b \leq R/M$. Let $f : bR \to R/M$ be defined by $f(br) = r + M$. Note that $f$ is well-defined. So $1 + M = f(b) = cb + M + x$, $c \in R, x \in X_b, 1 - cb + M = x \in R/M \cap X_b = 0, 1 - cb \in M, cb \in RbR \subseteq M$, so $1 \in M$, which is a contradiction. This proves that $r(b) + RbR = R$, and hence $b = bd$ for some $d \in RbR \subseteq J(R)$. This implies $b = 0$, which is required contradiction.

Theorem 2.13 If $R$ is a ring whose every simple singular right $R$-module is right AP-injective, then $J(R) \cap Z(R) = 0$.

Proof. Take any $b \in J(R) \cap Z(R)$. If $b \neq 0$, then $r(b) \neq R$, and $r(b) + RbR$ is an essential right ideal of $R$. We will prove $r(b) + RbR = R$. If not, as the proof in Theorem 2.6, there exists a maximal essential right ideal $M$ of $R$ containing $r(b) + RbR$, so $R/M$ is AP-injective, thus $l_{R/M}r_{R}(b) = (R/M)b + X_b, X_b \leq R/M$. Let $f : bR \to R/M$ be defined by $f(br) = r + M$, note that $f$ is well-defined, $1 + M = f(b) = cb + M + x, c \in R, x \in X_b, 1 - cb + M = x \in R/M \cap X_b = 0, 1 - cb \in M, cb \in RbR \subseteq M$, so $1 \in M$, which is a contradiction. This proves that $r(b) + RbR = R$, and hence $b = bd$ for some $d \in RbR \subseteq J(R)$. This implies $b = 0$, which is a contradiction.
Theorem 2.14 If $R$ is a ring whose every simple singular right $R$–module is AP-injective, then $J(R)$ contains no nonzero nilpotent elements if and only if $J(R) = 0$.

Proof. Assume $J(R)$ contains nonzero nilpotent elements. Write $L = bR + r(b)$ for any $b \in J(R)$. If $L = R$, as the proof in [3, Proposition 2.11], $J(R) = 0$. If $L \neq R$, then there exists a right ideal $K$ of $R$ such that $L \oplus K$ is an essential right ideal of $R$. We claim that $L \oplus K = R$. If not, there exists a maximal essential right ideal $M$ of $R$ containing $L \oplus K$. By assumption, the simple singular right $R$–module $R/M$ is AP-injective, thus $l_{R/M}r_{R}(b) = (R/M)b \oplus X_b$, $X_b \subseteq R/M$. Let $f : bR \rightarrow R/M$ be defined by $f(br) = r + M$. Note that $f$ is well-defined. Thus there exists $c \in R$, $x \in X_b$ such that $1 + M = f(b) = cb + M + x$, then $1 - cb + M = x \in R/M \cap X_b = 0$, $1 - cb \in M$, Note that $cb \in J(R) \subseteq M$, so $1 \in M$, which is a contradiction. This shows that $L \oplus K = R$. Then $bR + r(b) = eR$, $e^2 = e \in R$, so $b^2 = beb = bab^2$ for some $a \in R$. But $b \in J(R)$, thus as the proof in [3, Proposition 2.11], $b = 0$. This gives that $J(R) = 0$. The converse is obvious.

Theorem 2.15 Let $R$ be a ring whose simple singular right $R$–modules are AP-injective. If $R$ is semiprime or abelian, then for any $a \in R$, $RaR + r(a) = R$. In particular, $a \in aRaR$ (equivalently, $R$ is right weakly regular), and $J(R) = 0$.

Proof. First as the proof in [12, Lemma 4.1], if $RaR + r(a) \subseteq M$ for some maximal right ideal of $R$, then $M$ is essential.

Next we will show if $0 \neq a \in R$ with $a^2 = 0$, then $RaR + r(a) = R$. If not, then there exists a maximal right ideal $M$ of $R$ such that $RaR + r(a) \subseteq M$, and so $M$ is essential, by the first part of the proof. Hence $R/M$ is right AP-injective, then as the proof in Theorem 2.3, $RaR + r(a) = R$. Thus $aRaR = aR$. If $0 \neq a \in J(R)$, $a = ab$, $b \in RaR \subseteq J(R)$, since $1 - b$ is invertible, $a = 0$. So $J(R) = 0$.

References


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