Direct Product of Fuzzy Groups and Fuzzy Rings

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Abstract

In this paper, we define direct product of fuzzy group and fuzzy ring by using the definition of Yuan and Lee’s fuzzy binary operation. Considering the structure of fuzzy group in [7], we observe that if $H_1 \times H_2$ is a fuzzy subgroup of $G \times G$ then both $H_1$ and $H_2$ are fuzzy subgroups of $G$ whereas either $H_1$ or $H_2$ is fuzzy subgroup of $G$ in Malik and Mordeson’s paper[4]. We obtain similar results for rings analogous to ordinary ring theory.

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1 Introduction

The concept of fuzzy sets was introduced by Zadeh [6], and the notion of fuzzy subgroups was defined and established by Rosenfeld [1]. Then the notion of fuzzy ideal of a ring was introduced by Liu [8]. Since then many mathematicians such as Aktaş and Çağman [5], Malik and Mordeson [3,4], Yuan and Lee [7] have studied about it. Fuzzy relations on set were first introduced by Zadeh [6]. Fuzzy relations on rings and groups were studied by Malik and Mordeson [3,4] and Ersoy [2] generalized them.

In this study, using Yuan and Lee’s definition of fuzzy group based on fuzzy binary operation and Aktaş and Çağman’s definition of fuzzy ring, we investigate direct product of fuzzy groups and fuzzy rings. By using our
definition, we obtain some results which are different from Malik and Morde-
son’s [4] paper. That is; $H_1 \times H_2$ is a fuzzy subgroup of $G \times G$ if and only if 
both $H_1$ and $H_2$ are fuzzy subgroups of $G$. $I_1 \times I_2$ is a fuzzy ideal of $R_1 \times R_2$
if and only if both $I_1$ and $I_2$ are fuzzy ideals of $R_1$ and $R_2$.

2 Preliminaries

In this section, we shall formulate the preliminary definitions and results that 
are required later in this paper.

Definition 2.1 [7] Let $G$ be a nonempty set and $R$ be a fuzzy subset of 
$G \times G \times G$. $R$ is called a fuzzy binary operation on $G$ if

1. $\forall a, b \in G$, $\exists c \in G$ such that $R(a, b, c) > \theta$;
2. $\forall a, b, c_1, c_2 \in G$, $R(a, b, c_1) > \theta$ and $R(a, b, c_2) > \theta$ implies $c_1 = c_2$.

Let $R$ be a fuzzy binary operation on $G$, then we have a mapping

$$R : F(G) \times F(G) \rightarrow F(G)$$

\[(A, B) \mapsto R(A, B)\]

where $F(G) = \{A \mid A : G \rightarrow [0, 1] \text{ is a mapping}\}$ and

$$R(A, B)(c) = \bigvee_{a, b \in G} (A(a) \land B(b) \land R(a, b, c)) \quad (1)$$

Let $A = \{a\}$ ve $B = \{b\}$ and let $R(A, B)$ be denoted as $a \circ b$, then

$$(a \circ b)(c) = R(a, b, c), \forall c \in G \quad (2)$$

$$((a \circ b) \circ c)(z) = \bigvee_{d \in G} (R(a, b, d) \land R(d, c, z)) \quad (3)$$

$$(a \circ (b \circ c))(z) = \bigvee_{d \in G} (R(b, c, d) \land R(a, d, z)) \quad (4)$$

Definition 2.2 [7] Let $G$ be a nonempty set and $R$ be a fuzzy binary op-
eration on $G$. $(G, R)$ is called a fuzzy group if the following conditions are 
true:

G1 $\forall a, b, c, z_1, z_2 \in G$, $((a \circ b) \circ c)(z_1) > \theta$ and $(a \circ (b \circ c))(z_2) > \theta$ implies 
$z_1 = z_2$;

G2 $\exists e_o \in G$ such that $(e_o \circ a)(a) > \theta$ and $(a \circ e_o)(a) > \theta$ for any $a \in G$. $(e_o$ 
is called an identity element of $G$);
G3 \( \forall a \in G, \exists b \in G \text{ such that } (a \circ b)(e_o) > \theta \text{ and } (b \circ a)(e_o) > \theta. \) (b is called an inverse element of a and denoted by \( a^{-1} \)).

**Lemma 2.3** \[7\] H is a fuzzy subgroup of G if and only if
1. \( \forall a, b \in H , \forall c \in G, \ (a \circ b)(c) > \theta \implies c \in H ; \)
2. \( a \in H \implies a^{-1} \in H. \)

**Definition 2.4** \[5\] Let \( (G, R) \) be a fuzzy subgroup. If

\[ (a \circ b)(c) > \theta \iff (b \circ a)(c) > \theta, \quad \forall a, b, c \in G, \]

then \( (G, R) \) is called abelian fuzzy group.

**Definition 2.5** \[7\] Let \( (G_1, R_1) \) and \( (G_2, R_2) \) be two fuzzy groups and \( f : G_1 \rightarrow G_2 \) be a mapping. If

\[ R_1(a, b, c) > \theta \Rightarrow R_2(f(a), f(b), f(c)) > \theta \]

then \( f \) is called a fuzzy (group) homomorphism.

Let \( G \) be a fuzzy binary operation on \( R \). Then we have a mapping

\[
G : F(R) \times F(R) \rightarrow F(R) \\
(A, B) \mapsto G(A, B)
\]

where \( F(R) = \{ A \mid A : R \rightarrow [0, 1] \text{ is a mapping} \} \) and

\[
G(A, B)(c) = \bigvee_{a, b \in R} (A(a) \land B(b) \land G(a, b, c)) \tag{5}
\]

Let \( A = \{ a \} \) and \( B = \{ b \} \) and let \( G(A, B) \) and \( H(A, B) \) be denoted as \( a \circ b \) and \( a \ast b \), respectively. Then

\[
(a \circ b)(c) = G(a, b, c), \forall c \in R \tag{6}
\]
\[
(a \ast b)(c) = H(a, b, c), \forall c \in R \tag{7}
\]
\[
((a \circ b) \circ c)(z) = \bigvee_{d \in R} (G(a, b, d) \land G(d, c, z)) \tag{8}
\]
\[
(a \circ (b \circ c))(z) = \bigvee_{d \in R} (G(b, c, d) \land G(a, d, z)) \tag{9}
\]
\[
(a \ast (b \circ c))(z) = \bigvee_{d \in R} (G(b, c, d) \land H(a, d, z)) \tag{10}
\]
\[
((a \ast b) \circ (a \ast c))(z) = \bigvee_{d, e \in R} (H(a, b, d) \land H(a, c, e) \land G(d, e, z)) \tag{11}
\]
Definition 2.6 [5] Let $R$ be a nonempty set and $G$ and $H$ be two fuzzy binary operations on $R$. Then $(R, G, H)$ is called fuzzy ring if the following conditions hold.

R1 $(R, G)$ is an abelian fuzzy group;

R2 $\forall a, b, c, z_1, z_2 \in R$, $(a * b * c)(z_1) > \theta$ and $(a * (b * c))(z_2) > \theta$ implies $z_1 = z_2$;

R3 $\forall a, b, c, z_1, z_2 \in R$, $((a \circ b) * c)(z_1) > \theta$ and $((a \circ c) \circ (b * c))(z_2) > \theta$ implies $z_1 = z_2$.

Definition 2.7 [5] Let $(R, G, H)$ be a fuzzy ring.

1 If $(a * b)(u) > \theta \Leftrightarrow (b * a)(u) > \theta$, then $(R, G, H)$ is said to be a commutative fuzzy ring.

2 If $\exists e_+ \in R$ such that $(a * e_+)(a) > \theta$ and $(e_+ * a)(a) > \theta$ for every $a \in R$ then $(R, G, H)$ is said to be a fuzzy ring with identity.

3 Let $(R, G, H)$ be a fuzzy ring with identity. If $(a * b)(e_+) > \theta$ and $(b * a)(e_+) > \theta$, $\forall a \in R$, $\exists b \in R$ then $b$ is said to be an inverse element of $a$ and is denoted by $a^{-1}$.

Definition 2.8 [5] A nonempty subset $I$ of a fuzzy ring $(R, G, H)$ is called a fuzzy ideal of $R$ if the following conditions are satisfied.

1 $\forall x, y \in I$, $(x \circ y)(z) > \theta \Rightarrow z \in I$ for all $z \in R$

2 $\forall x \in I$, $x^{-1} \in I$

3 For all $s \in I$, for all $r \in R$, $(r * s)(x) > \theta \Rightarrow x \in I$ and $(s * r)(y) > \theta \Rightarrow y \in I$, $x, y \in R$.

3 Direct Product of Fuzzy Groups

Let $(G, J_G)$ and $(H, J_H)$ be two fuzzy groups and $S$ be a fuzzy binary operation on $G \times H$. Then we have a mapping

$$S : F(G \times H) \times F(G \times H) \rightarrow F(G \times H)$$

$$(A_1 \times B_1, A_2 \times B_2) \mapsto S(A_1 \times B_1, A_2 \times B_2)$$

where $F(G \times H) = \{A \times B \mid A \times B : G \times H \rightarrow [0, 1] \text{ is a mapping, } A \text{ and } B \text{ are fuzzy subsets of } G \text{ and } H, \text{ respectively}\}$ and

$$S(A_1 \times B_1, A_2 \times B_2)(x, y) = \bigvee_{x_1, x_2 \in G, y_1, y_2 \in H} \left( A_1(x_1) \land A_2(x_2) \land B_1(y_1) \land B_2(y_2) \land J_G(x_1, x_2, x) \land J_H(y_1, y_2, y) \right) \quad (12)$$
Let $A_1 \times B_1 = \{(x_1, y_1)\}$ and $A_2 \times B_2 = \{(x_2, y_2)\}$, and $S(A_1 \times B_1, A_2 \times B_2)$ be denoted as $(x_1, y_1) \oplus (x_2, y_2)$, then

\[
((x_1, y_1) \oplus (x_2, y_2))(x, y) = J_G(x_1, x_2, x) \land J_H(y_1, y_2, y) \tag{13}
\]

\[
(((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3))(x, y) = \bigvee_{x_4 \in G, y_4 \in H} \left( J_G(x_1, x_2, x_4) \land J_H(y_1, y_2, y_4) \wedge (J_G(x_4, x_3, x) \land J_H(y_4, y_3, y)) \right)
\]

\[
((x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)))(x, y) = \bigvee_{x_4 \in G, y_4 \in H} \left( J_G(x_2, x_3, x_4) \land J_H(y_2, y_3, y_4) \wedge (J_G(x_1, x_4, x) \land J_H(y_1, y_4, y)) \right)
\]

**Theorem 3.1** $(G \times H, S)$ is a fuzzy group.

**Proof** Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (u_1, v_1), (u_2, v_2) \in G \times H$ such that

\[
(((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3))(u_1, v_1) > \theta,
((x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)))(u_2, v_2) > \theta.
\]

Then there exists $x, x' \in G$ and $y, y' \in H$ such that

\[
J_G(x_1, x_2, x) \land J_H(y_1, y_2, y) \land J_G(x, x_3, u_1) \land J_H(y, y_3, v_1) > \theta,
J_G(x_2, x_3, x') \land J_H(y_2, y_3, y') \land J_G(x_1, x', v_2) \land J_H(y_1, y', v_2) > \theta.
\]

Since $G$ and $H$ are fuzzy groups, we have $u_1 = u_2$, $v_1 = v_2$ and so $(u_1, v_1) = (u_2, v_2)$. Let $(x, y) \in G \times H$, and let $e_1$ and $e_2$ be identity elements of fuzzy groups $G_1$ and $G_2$, respectively. By

\[
((x, y) \oplus (e_1, e_2))(x, y) > J_G(x, e_1, x) \land J_H(y, e_2, y) > \theta,
((e_1, e_2) \oplus (x, y))(x, y) > J_G(e_1, x, x) \land J_H(e_2, y, y) > \theta.
\]

Thus $(e_1, e_2) \in G \times H$. Let $(x, y) \in G \times H$ and let $x^{-1}$ and $y^{-1}$ be inverse elements of $a$ in $G$ and $b$ in $H$, respectively. By

\[
((x, y) \oplus (x^{-1}, y^{-1}))(e_1, e_2) > J_G(x, x^{-1}, e_1) \land J_H(y, y^{-1}, e_2) > \theta,
((x^{-1}, y^{-1}) \oplus (x, y))(e_1, e_2) > J_G(x^{-1}, x, e_1) \land J_H(y^{-1}, y, e_2) > \theta,
\]

we have $(x, y)^{-1} = (x^{-1}, y^{-1}) \in G \times H$. Consequently, $(G \times H, S)$ is a fuzzy group from definition 2.2.

**Theorem 3.2** Let $(G_1, J_1)$ and $(G_2, J_2)$ be two fuzzy groups and let $H_1$ and $H_2$ be two nonempty subsets of $G_1$ and $G_2$, respectively. Then $H_1 \times H_2$ is a fuzzy subgroup of $G_1 \times G_2$ if and only if both $H_1$ and $H_2$ are fuzzy subgroups of $G_1$ and $G_2$. 


Proof Let $H_1 \times H_2$ is a fuzzy subgroup of $G_1 \times G_2$ and $(x_1, y_1), (x_2, y_2) \in H_1 \times H_2$. Then there exists $(x, y) \in G_1 \times G_2$ such that $S((x_1, y_1), (x_2, y_2), (x, y)) > \theta$. By
\[
((x_1, y_1) \oplus (x_2, y_2))(x, y) = J_1(x_1, x_2, x) \land J_2(y_1, y_2, y) > \theta,
\]
we have $x \in H_1$ and $y \in H_2$, therefore $(x, y) \in H_1 \times H_2$.

For $(x, y) \in H_1 \times H_2$, there exists $(u, v) \in H_1 \times H_2$ such that $S((x, y), (u, v), (e_1, e_2)) > \theta$ and $S((u, v), (x, y), (e_1, e_2)) > \theta$. By
\[
((x, y) \oplus (u, v))(e_1, e_2) > J_1(x, u, e_1) \land J_2(y, v, e_2) > \theta,
\]
we have $x^{-1} = u \in H_1$ and $y^{-1} = v \in H_2$. Consequently, from Proposition 2.3, $H_1$ and $H_2$ are fuzzy subgroups of $G_1$ and $G_2$, respectively.

Conversely, let $H_1$ and $H_2$ are fuzzy subgroups of $G_1$ and $G_2$. Hence $(H_1 \times H_2, S)$ is a fuzzy subgroup of $(G_1 \times G_2, S)$ according to Theorem 3.1.

Corollary 3.3 Let $(G, J)$ be a fuzzy group and $H$ be a nonempty subset of $G$. $H \times H$ is a fuzzy subgroup of $G \times G$ if and only if $H$ is a fuzzy subgroup of $G$.

Corollary 3.4 Let $A_1, A_2, \ldots, A_n$ be nonempty subsets of fuzzy groups $G_1, G_2, \ldots, G_n$, respectively. $A_1 \times A_2 \times \ldots \times A_n$ is a fuzzy subgroup of $G_1 \times G_2 \times \ldots \times G_n$ if and only if $A_i$ is a fuzzy subgroup of $G_i$, for all $i \in \{1, 2, \ldots, n\}$.

Theorem 3.5 $(G_1, J_1)$ and $(G_2, J_2)$ are two abelian fuzzy groups if and only if $(G_1 \times G_2, S)$ is an abelian fuzzy group.

Proof Let $(x_1, x_2), (y_1, y_2), (u_1, u_2), (v_1, v_2) \in G_1 \times G_2$ such that $((x_1, x_2) \oplus (y_1, y_2))(u_1, u_2) > \theta$ and $((y_1, y_2) \oplus (x_1, x_2))(v_1, v_2) > \theta$. Then
\[
J_1(x_1, y_1, u_1) \land J_2(x_2, y_2, u_2) > \theta,
\]
\[
J_1(y_1, x_1, v_1) \land J_2(y_2, x_2, v_2) > \theta.
\]

Since $G_1$ and $G_2$ are abelian, $u_1 = v_1$ and $u_2 = v_2$, so that $(u_1, u_2) = (v_1, v_2)$. Conversely, suppose $x, y, u, v \in G_1$, such that $J(x, y, u) > \theta$ and $J(y, x, v) > \theta$. Since
\[
((x, e_2) \oplus (y, e_2))(u, e_2) > J_1(x, y, u) \land J_2(e_2, e_2, e_2) > \theta,
\]
\[
((y, e_2) \oplus (x, e_2))(v, e_2) > J_1(y, x, v) \land J_2(e_2, e_2, e_2) > \theta,
\]
and $G_1 \times G_2$ are abelian, we have $(u, e_2) = (v, e_2)$ and so $u = v$. Thus $G_1$ is an abelian fuzzy group. Similarly, it is clear that $G_2$ is also an abelian fuzzy group.
Theorem 3.6 Let \( f : G_1 \times G_2 \rightarrow H_1 \times H_2 \) be a fuzzy group homomorphism. Then

1 If \( A \times B \) is a fuzzy subgroup of \( G_1 \times G_2 \) then \( f(A \times B) \) is a fuzzy subgroup of \( H_1 \times H_2 \).

2 If \( A \times B \) is a fuzzy subgroup of \( H_1 \times H_2 \) then \( f^{-1}(A \times B) \) is a fuzzy subgroup of \( G_1 \times G_2 \).

Proof 1 Let \((x_1, x_2), (y_1, y_2) \in f(A \times B)\). Then, there exists \((a_1, a_2), (b_1, b_2) \in A \times B\) such that \((x_1, x_2) = f(a_1, a_2)\) and \((y_1, y_2) = f(b_1, b_2)\). Let \((c_1, c_2) \in A \times B\) such that \(((a_1, a_2) \oplus (b_1, b_2))(c_1, c_2) > \theta\). Since \( f \) is a fuzzy homomorphism, we get \(((x_1, x_2) \oplus (y_1, y_2))(f(c_1, c_2)) > \theta\), \((c_1, c_2) \in f(A \times B)\).

Let \((e_1, e_2)\) be identity element of \( A \times B\). Hence, for \((a_1, a_2) \in A \times B\), \(((a_1, a_2) \oplus (e_1, e_2))(a_1, a_2) > \theta\) and \(((e_1, e_2) \oplus (a_1, a_2))(a_1, a_2) > \theta\). So

\[
((x_1, x_2) \oplus f(e_1, e_2))(x_1, x_2) > \theta,
\]

\[
(f(e_1, e_2) \oplus (x_1, x_2))(x_1, x_2) > \theta.
\]

Thus \( f(e_1, e_2) \) is identity element of \( f(A \times B) \).

Let \((u, v)\) be inverse element of \((a_1, a_2)\) in fuzzy group \( G_1 \times G_2 \). Then \(((a_1, a_2) \oplus (u, v))(e_1, e_2) > \theta\) and \(((u, v) \oplus (a_1, a_2))(e_1, e_2) > \theta\). Using the fact that \( f \) is a fuzzy group homomorphism, we deduce that

\[
((x_1, x_2) \oplus f(u, v))(f(e_1, e_2)) > \theta,
\]

\[
(f(u, v) \oplus (x_1, x_2))(f(e_1, e_2)) > \theta.
\]

It follows that \((x_1, x_2)^{-1} = f(u, v) \in f(A \times B)\). Therefore, by Proposition 2.3, \( f(A \times B) \) is a fuzzy subgroup of \( H_1 \times H_2 \).

2 Similar to part 1.

Corollary 3.7 Let \( f : G_1 \times G_2 \times \ldots \times G_n \rightarrow H_1 \times H_2 \times \ldots \times H_n \) be a fuzzy group homomorphism. Then

1 If \( A_1 \times A_2 \times \ldots \times A_n \) is a fuzzy subgroup of \( G_1 \times G_2 \times \ldots \times G_n \) then \( f(A_1 \times A_2 \times \ldots \times A_n) \) is a fuzzy subgroup of \( H_1 \times H_2 \times \ldots \times H_n \);

2 If \( A_1 \times A_2 \times \ldots \times A_n \) is a fuzzy subgroup of \( H_1 \times H_2 \times \ldots \times H_n \) then \( f^{-1}(A_1 \times A_2 \times \ldots \times A_n) \) is a fuzzy subgroup of \( G_1 \times G_2 \times \ldots \times G_n \).

Proof It can be proved easily by mathematical induction method.

Similar to direct product of two fuzzy groups, now we can define direct product of two fuzzy rings as follows.
4 Direct Product of Fuzzy Rings

Let \((R_1, G_1, H_1)\), \((R_2, G_2, H_2)\) be two fuzzy rings and \(S_1\), \(S_2\) be two fuzzy binary operations on \(R_1 \times R_2\). Then we have a mapping

\[
S_1 : F(R_1 \times R_2) \times F(R_1 \times R_2) \rightarrow F(R_1 \times R_2) \\
(A_1 \times B_1, A_2 \times B_2) \mapsto S_1(A_1 \times B_1, A_2 \times B_2)
\]

where \(F(R_1 \times R_2) = \{A \times B \mid A \times B : R_1 \times R_2 \rightarrow [0, 1]\} is a mapping, \(A\) and \(B\) are fuzzy subsets of \(R_1\) and \(R_2\), respectively\} and

\[
S_1(A_1 \times B_1, A_2 \times B_2)(x, y) = \bigvee_{x_1, x_2 \in R_1 \atop y_1, y_2 \in R_2} \left( A_1(x_1) \wedge A_2(x_2) \wedge B_1(y_1) \wedge B_2(y_2) \right) \\
\wedge G_1(x_1, x_2, x) \wedge G_2(y_1, y_2, y)
\]

(14)

and

\[
S_2 : F(R_1 \times R_2) \times F(R_1 \times R_2) \rightarrow F(R_1 \times R_2) \\
(A_1 \times B_1, A_2 \times B_2) \mapsto S_2(A_1 \times B_1, A_2 \times B_2)
\]

\[
S_2(A_1 \times B_1, A_2 \times B_2)(x, y) = \bigvee_{x_1, x_2 \in R_1 \atop y_1, y_2 \in R_2} \left( A_1(x_1) \wedge A_2(x_2) \wedge B_1(y_1) \wedge B_2(y_2) \right) \\
\wedge H_1(x_1, x_2, x) \wedge H_2(y_1, y_2, y)
\]

(15)

Let \(A_1 \times B_1 = \{(x_1, y_1)\}\) and \(A_2 \times B_2 = \{(x_2, y_2)\}\, and \, S_1(A_1 \times B_1, A_2 \times B_2)\) and \(S_2(A_1 \times B_1, A_2 \times B_2)\) be denoted as \((x_1, y_1) \oplus (x_2, y_2)\) and \((x_1, y_1) \odot (x_2, y_2)\) then

\[
((x_1, y_1) \oplus (x_2, y_2))(x, y) = (G_1(x_1, x_2, x) \wedge G_2(y_1, y_2, y)) \\
((x_1, y_1) \odot (x_2, y_2))(x, y) = (H_1(x_1, x_2, x) \wedge H_2(y_1, y_2, y))
\]

\[
((x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)))(x, y) = \bigvee_{x_4 \in R_1 \atop y_4 \in R_2} \left( G_1(x_2, x_3, x_4) \wedge G_2(y_2, y_3, y_4) \right) \\
\wedge G_1(x_1, x_4, x) \wedge G_2(y_1, y_4, y)
\]

\[
((x_1, y_1) \odot ((x_2, y_2) \odot (x_3, y_3)))(x, y) = \bigvee_{x_4 \in R_1 \atop y_4 \in R_2} \left( H_1(x_2, x_3, x_4) \wedge H_2(y_2, y_3, y_4) \right) \\
\wedge H_1(x_1, x_4, x) \wedge H_2(y_1, y_4, y)
\]

\[
((x_1, y_1) \odot ((x_2, y_2) \oplus (x_3, y_3)))(x, y) = \bigvee_{x_4 \in R_1 \atop y_4 \in R_2} \left( G_1(x_2, x_3, x_4) \wedge G_2(y_2, y_3, y_4) \right) \\
\wedge H_1(x_1, x_4, x) \wedge H_2(y_1, y_4, y)
\]

\[
(((x_1, y_1) \odot (x_2, y_2)) \oplus ((x_1, y_1) \odot (x_3, y_3))(x, y) = \bigvee_{x_4 \in R_1 \atop y_4 \in R_2} \left( H_1(x_1, x_2, x_4) \wedge H_2(y_1, y_2, y_4) \right) \\
\wedge H_1(x_1, x_3, x_5) \wedge H_2(y_1, y_3, y_5) \wedge G_1(x_4, x_5, x) \wedge G_2(y_4, y_5, y)
\]
Theorem 4.1 \((R_1 \times R_2, S_1, S_2)\) is a fuzzy ring.

**Proof** Clearly \((R_1 \times R_2, S_1)\) is an abelian fuzzy group by Theorem 3.1 and Theorem 3.5. Similar to the proof of Theorem 3.1, it is easy to show that the condition R2 in Definition 2.6 is satisfied. Hence, it suffices to show R3. For this purpose, let \((x_1, y_1), (x_2, y_2), (x_3, y_3), (u_1, v_1), (u_2, v_2) \in R_1 \times R_2\) such that \(((x_1, y_1) \otimes ((x_2, y_2) \oplus (x_3, y_3)))(u_1, v_1) \geq \theta\) and \(((x_1, y_1) \otimes (x_2, y_2)) \oplus ((x_1, y_1) \otimes (x_3, y_3))(u_2, v_2) \geq \theta\). Then there exists \(x_4, x_5, x_6 \in R_1\) and \(y_4, y_5, y_6 \in R_2\) such that

\[
G_1(x_2, x_3, x_4) \land G_2(y_2, y_3, y_4) \land H_1(x_1, x_4, u_1) \land H_2(y_1, y_4, v_1) \geq \theta,
\]

\[
H_1(x_1, x_2, x_5) \land H_2(y_1, y_2, y_5) \land H_1(x_1, x_3, x_6) \land H_2(y_1, y_3, y_6) \land G_1(x_5, x_6, u_2) \land G_2(y_5, y_6, v_2) \geq \theta.
\]

Since \((R_1, G_1, H_1)\) and \((R_2, G_2, H_2)\) be two fuzzy rings, \(u_1 = u_2\), \(v_1 = v_2\) and consequently \((u_1, v_1) = (u_2, v_2)\). The result now follows.

Theorem 4.2 Let \((R_1, G_1, H_1)\) and \((R_2, G_2, H_2)\) be two fuzzy rings and let \(I_1\) and \(I_2\) be two nonempty subsets of \(R_1\) and \(R_2\), respectively. Then \(I_1 \times I_2\) is a fuzzy ideal of \(R_1 \times R_2\) if and only if both \(I_1\) and \(I_2\) are fuzzy ideals of \(R_1\) and \(R_2\).

**Proof** Let \(I_1 \times I_2\) be a fuzzy ideal of \(R_1 \times R_2\). Then \((I_1 \times I_2, S_1)\) is a fuzzy subgroup of \((R_1 \times R_2, S_1)\). By Theorem 3.2, \((I_1, G_1)\) and \((I_2, G_2)\) are fuzzy subgroups of \((R_1, G_1)\) and \((R_2, G_2)\), respectively. Let \(r_1, y_1 \in R_1\), \(r_2, y_2 \in R_2\), \(x_1 \in I_1\), \(x_2 \in I_2\) such that \(H_1(r_1, x_1, y_1) \geq \theta\) and \(H(r_2, x_2, y_2) \geq \theta\). Then

\[
((r_1, r_2) \otimes (x_1, x_2))(y_1, y_2) = H_1(r_1, x_1, y_1) \land H_2(r_2, x_2, y_2) \geq \theta.
\]

Hence \((y_1, y_2) \in I_1 \times I_2\), and consequently \(y_1 \in I_1\) and \(y_2 \in I_2\).

Clearly we have the following.

**Corollary 4.3** Let \((R, G, H)\) be a fuzzy ring and \(I\) be a nonempty subset of \(R\). \(I \times I\) is a fuzzy ideal of \(R \times R\) if and only if \(I\) is fuzzy ideal of \(R\).

**Corollary 4.4** Let \(I_1, I_2, \ldots, I_n\) be nonempty subsets of fuzzy rings \(R_1, R_2, \ldots, R_n\), respectively. \(I_1 \times I_2 \times \ldots \times I_n\) is a fuzzy ideals of \(R_1 \times R_2 \times \ldots \times R_n\) if and only if \(I_i\) is a fuzzy ideal of \(R_i\) for all \(i \in \{1, 2, \ldots, n\}\).

Theorem 4.5 Let \((R, G, H)\) be a fuzzy ring. Then

1. \(R \times R\) is an abelian ring if and only if \(R\) is an abelian ring,
2. \(R \times R\) has an identity element if and only if \(R\) has an identity element.

**Proof** It can be proved similar to Theorem 3.1 and Theorem 3.5.
Theorem 4.6 Let $f : R_1 \times R_2 \rightarrow K_1 \times K_2$ be a fuzzy ring homomorphism. Then

1. If $f$ is surjective and $I_1 \times I_2$ is a fuzzy ideal of $R_1 \times R_2$ then $f(I_1 \times I_2)$ is a fuzzy ideal of $K_1 \times K_2$.

2. If $I_1 \times I_2$ is a fuzzy ideal of $K_1 \times K_2$ then $f^{-1}(I_1 \times I_2)$ is a fuzzy ideal of $R_1 \times R_2$.

Proof 1 Let $I_1 \times I_2$ is a fuzzy ideal of $R_1 \times R_2$. Then $I_1 \times I_2$ is a fuzzy subgroup of $R_1 \times R_2$. By Theorem 3.6, $f(I_1 \times I_2)$ is fuzzy subgroup of $K_1 \times K_2$. Let $(k_1, k_2) \in K_1 \times K_2$ and $(x, y) \in f(I_1 \times I_2)$. Then there exists $(r_1, r_2) \in R_1 \times R_2$ and $(a, b) \in I_1 \times I_2$ such that $(k_1, k_2) = f(r_1, r_2)$ and $(x, y) = f(a, b)$. Let $(c, d) \in I_1 \times I_2$ such that $((r_1, r_2) \otimes (a, b))(c, d) > \theta$. Since $I_1 \times I_2$ is a fuzzy ideal, we have $(c, d) \in I_1 \times I_2$. Since $f$ is a fuzzy homomorphism, we have $((k_1, k_2) \otimes (x, y))(f(c, d)) > \theta$, and $f(c, d) \in f(I_1 \times I_2)$.

2. Similar to part 1.

Corollary 4.7 Let $f : R_1 \times R_2 \times \ldots \times R_n \rightarrow K_1 \times K_2 \times \ldots \times K_n$ be a fuzzy ring homomorphism. Then

1. If $f$ is surjective and $I_1 \times I_2 \times \ldots \times I_n$ is a fuzzy ideal of $R_1 \times R_2 \times \ldots \times R_n$ then $f(I_1 \times I_2 \times \ldots \times I_n)$ is fuzzy ideal of $K_1 \times K_2 \times \ldots \times K_n$.

2. If $I_1 \times I_2 \times \ldots \times I_n$ is a fuzzy ideal of $K_1 \times K_2 \times \ldots \times K_n$ then $f^{-1}(I_1 \times I_2 \times \ldots \times I_n)$ is fuzzy ideal of $R_1 \times R_2 \times \ldots \times R_n$.

Proof It can be easily shown by mathematical induction method.

References


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