On a Certain Class of Translation Surfaces in a Pseudo-Galilean Space

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Abstract

In this paper we describe a special class of translation surfaces in a pseudo-Galilean space. We analyze translation surfaces having constant Gaussian and mean curvatures, as well as translation Weingarten surfaces.

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1 Introduction

In this paper we describe a special class of translation surfaces in a pseudo-Galilean space. We are specially interested in the analogues of the results from the Euclidean space concerning translation surfaces having constant Gaussian curvature $K$ and mean curvature $H$, and translation surfaces that are Weingarten surfaces as well.

A translation surface is a surface that can locally be written as the sum of two curves $\alpha, \beta$

$$\mathbf{x}(u, v) = \alpha(u) + \beta(v).$$

Translation surfaces in the Euclidean and Minkowski space having constant Gaussian and mean curvature are described in [8]:

Theorem 1.1 Let $S$ be a translation surface with constant Gaussian curvature $K$ in 3-dimensional Euclidean space or 3-dimensional Minkowski space. Then $S$ is congruent to a cylindrical surface (i.e. generalized cylinder), so $K = 0$. 

Theorem 1.2 Let $S$ be a translation surface with constant mean curvature $H \neq 0$ in 3-dimensional Euclidean space. Then $S$ is congruent to the following surface or a part of it (class of cylindrical surfaces)

$$
z = -\frac{\sqrt{1+a^2}}{2H}\sqrt{1-4H^2x^2} - ay, \quad a \in \mathbb{R},
$$

and for $H = 0$ to a plane or to the Scherk minimal surface

$$
z = \frac{1}{a}\log(\cos(ax)) - \frac{1}{a}\log(\cos(ay)), \quad a \in \mathbb{R} \setminus \{0\}.
$$

For the translation Weingarten surfaces in Euclidean space the following theorem holds ([1]):

Theorem 1.3 A translation surface in $\mathbb{R}^3$ is a Weingarten surface if and only if it is either (a part of)

1. a plane,
2. a cylindrical surface,
3. the minimal surface of Scherk,
4. an orthogonal elliptic paraboloid parametrized by $x(s,t) = (s,t,a(s^2 + t^2))$.

Counterparts of these results for surfaces in Minkowski space can be found in [1]. In [9] the same problems were treated in a Galilean space.

2 Preliminary Notes

The pseudo-Galilean space $G^1_3$ is a Cayley-Klein space with absolute figure consisting of the ordered triple $\{\omega, f, I\}$, where $\omega$ is the ideal (absolute) plane in the real three-dimensional projective space $\mathcal{P}_3(\mathbb{R})$, $f$ the line (absolute line) in $\omega$ and $I$ the fixed hyperbolic involution of points of $f$. Homogeneous coordinates in $G^1_3$ are introduced in such a way that the absolute plane $\omega$ is given by $x_0 = 0$, the absolute line $f$ by $x_0 = x_1 = 0$ and the hyperbolic involution by $(0 : 0 : x_2 : x_3) \mapsto (0 : 0 : x_3 : x_2)$. The last condition is equivalent to the requirement that the conic $x_2^2 - x_3^2 = 0$ is the absolute conic. Metric relations are introduced with respect to the absolute figure. In affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, distance between points $P_i = (x_i, y_i, z_i)$, $i = 1, 2$, is defined by

$$
d(P_1, P_2) = \begin{cases} 
|x_2 - x_1|, & \text{if } x_1 \neq x_2, \\
\sqrt{|(y_2 - y_1)^2 - (z_2 - z_1)^2|}, & \text{if } x_1 = x_2.
\end{cases}
$$

(1)
The group of motions of $G^1_3$ is a six-parameter group given (in affine coordinates) by
\[
\begin{align*}
\bar{x} &= a + x \\
\bar{y} &= b + cx + y \cosh \varphi + z \sinh \varphi \\
\bar{z} &= d + ex + y \sinh \varphi + z \cosh \varphi.
\end{align*}
\] (2)

It leaves invariant the absolute figure as well the pseudo-Galilean distance (1) of points.

The pseudo-Galilean space $G^1_3$ can be also treated as a Cayley-Klein space equipped with the projective metric of signature $(0,0,+,-)$, as explained in [10]. According to the description of the Cayley-Klein spaces in [7], it is denoted by $P^3_{11001}$ and also called the Galilean space of index 1.

In the pseudo-Galilean space, a vector is called isotropic if it is of the form $(0,y,z)$. Among these vectors, there are also vectors with supplementary norm $\sqrt{|y^2 - z^2|}$ equal to zero, they are called lightlike vectors (vectors parallel to the planes $y = \pm z$). Isotropic vectors satisfying $y^2 - z^2 > 0$ are said to be spacelike vectors, and vectors satisfying $y^2 - z^2 < 0$ timelike vectors. In Figure 1 planes $y = \pm z$ and hyperbolic cylinders $y^2 - z^2 = \pm 1$ are presented.

A plane of the form $x = \text{const.}$ is called a pseudo-Euclidean plane (since its induced geometry is pseudo-Euclidean, i.e. Minkowski plane geometry), otherwise it is called isotropic (since its induced geometry is isotropic, i.e. Galilean plane geometry).

A $C^r$-surface, $r \geq 2$, is a subset $\Phi \subset G^1_3$ for which there exists an open subset $D$ of $\mathbb{R}^2$ and $C^r$-mapping $x : D \to G^1_3$ satisfying $\Phi = x(D)$. A $C^r$-surface $\Phi \subset G^1_3$ is called regular if $x$ is an immersion, and simple if $x$ is an embedding. It is admissible if it does not have pseudo-Euclidean tangent planes. If we denote $x = x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)$, $x_i = \frac{\partial x}{\partial u_i}$, $y_i = \frac{\partial y}{\partial u_i}$, $z_i = \frac{\partial z}{\partial u_i}$, then a surface is admissible if and only if $x_i \neq 0$, for some $i = 1, 2$.

Let $\Phi \subset G^1_3$ be a regular admissible surface. Then the unit normal vector field of a surface $x(u,v)$ is equal to
\[
N(u,v) = \frac{1}{W(u,v)}(0, x_1 z_2 - x_2 z_1, x_1 y_2 - x_2 y_1),
\]
\[
W(u,v) = \sqrt{|(x_1 y_2 - x_2 y_1)^2 - (x_1 z_2 - x_2 z_1)^2|}.
\] (3)

The function $W$ is equal to the pseudo-Galilean norm the vector $x_1 x_2 - x_2 x_1$. Vector defined by $\sigma = \frac{1}{W}(x_1 x_2 - x_2 x_1)$ is called a side tangential vector. We will not consider surfaces with $W = 0$, i.e. surfaces having lightlike side tangential vector (lightlike surfaces).
Since the normal vector field satisfies $N \cdot N = \epsilon = \pm 1$, we distinguish two basic types of admissible surfaces: spacelike surfaces having timelike surface normals ($\epsilon = -1$) and timelike surfaces having spacelike normals ($\epsilon = 1$). A surface is spacelike if $(x, y, z) = (x_1 y, y, 2 y, 1 - (x, 2 y, 1 - (x, 2 z, 1)) > 0$ in all of its points, timelike otherwise.

The first fundamental form of a surface is induced from the metric of the ambient space $G^1$

$$ds^2 = (g_1 du_1 + g_2 du_2)^2 + \delta(\tilde{x}_1 du_1 + \tilde{x}_2 du_2)^2,$$

where

$$\delta = \begin{cases} 
0, & \text{if direction } du_1 : du_2 \text{ is non-isotropic}, \\
1, & \text{if direction } du_1 : du_2 \text{ is isotropic}.
\end{cases}$$

By $\tilde{\cdot}$ above a vector, the projection of a vector on the pseudo-Euclidean $yz$-plane is denoted. The square on the second summand is the scalar square of the pseudo-Euclidean scalar product $(y_1, y_2) \cdot (z_1, z_2) = y_1 z_1 - y_2 z_2$.

We introduce the coefficients of the first fundamental form

$$g_i = x_{,i}, \quad h_{ij} = \tilde{x}_{,i} \cdot \tilde{x}_{,j}, \quad i, j = 1, 2.$$ 

Now we can write the first fundamental form as

$$ds^2 = (g_1 du_1 + g_2 du_2)^2 + \delta(h_{11} du_1^2 + 2 h_{12} du_1 du_2 + h_{22} du_2^2) = ds_1^2 + ds_2^2.$$ 

Notice that a surface is spacelike if $g_1^2 h_{22} - 2 g_1 g_2 h_{12} + g_2^2 h_{11} > 0$, timelike otherwise. Furthermore, the function $W^2$ satisfies

$$W^2 = -\epsilon (g_1 x_2 - g_2 x_1)^2$$

$$= -\epsilon (g_1^2 h_{22} - 2 g_1 g_2 h_{12} + g_2^2 h_{11}) > 0.$$ 

Therefore, notice that a spacelike surface has both parts of the first fundamental form, $ds_1^2$ and

$$ds_2^2 = -\epsilon \frac{W^2}{g_1^2} du_2^2, \quad g_1 \neq 0,$$ 

positive definite, while a timelike surfaces has the form $ds_1^2$ is positive definite and $ds_2^2$ negative definite. We have assumed here, without loss of generality, $g_1 \neq 0$.

The Gaussian curvature of a regular admissible surface is defined by

$$K = -\epsilon \frac{L_{11} L_{22} - L_{12}^2}{W^2}.$$
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where $L_{ij}$, $i, j = 1, 2$, are the coefficients of the second fundamental form

$$L_{ij} = \epsilon g_1 (g_{1,i,j} \bar{x} - g_{i,j} \bar{x}_1) \cdot N = \epsilon g_2 (g_{2,i,j} \bar{x} - g_{i,j} \bar{x}_2) \cdot N,$$

The mean curvature of a surface is defined by

$$H = -\frac{\epsilon}{2W^2} (g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}).$$

As in the Galilean space $G_3$ ([12]), the following theorem holds:

**Theorem 2.1** Minimal surfaces in a pseudo-Galilean space $G_3^1$ are ruled conoidal surfaces, i.e. they are cones with vertices on the absolute line or ruled surfaces with the absolute line as a director curve in infinity.

The geometry of the pseudo-Galilean space $G_3^1$ is developed in [2]. Some more results on surfaces in $G_3^1$ can be found in [3], [4], [5], [6].

3 Translation surfaces in the pseudo-Galilean space

We will consider translation surfaces in the pseudo-Galilean space $G_3^1$ that are obtained by translating two planar curves. In order to obtain an admissible surface, translated curves can be, with respect to the absolute figure, either an isotropic and a non-isotropic curve (obtained surfaces will be called translation surfaces of Type 1) or two non-isotropic curves (translation surfaces of Type 2). In this paper we shall treat the first ones, i.e. translation surfaces of Type 1.
These surfaces can be given by
\[ z = f(x) + g(y). \]  
They are obtained by translating a non-isotropic curve \( \alpha(x) = (x, 0, f(x)) \) along an isotropic curve \( \beta(y) = (0, y, g(y)) \) (or vice-versa). The curve \( \beta \) is a spacelike (resp. timelike) curve if and only if \( 1 - g^2(y) > 0 \) (resp. \( 1 - g^2(y) < 0 \)).

The side-tangential vector of a translation surface (4) is given by
\[ \sigma(x, y) = \frac{1}{W}(0, 1, g'(y)), \quad W = |1 - g'^2(y)| \]
and the unit normal field
\[ N(x, y) = \frac{1}{W}(0, g'(y), 1). \]

Since \( N^2 = g^2(y) - 1 \), the surface is a spacelike (resp. timelike) if and only if \( N^2 = g^2(y) - 1 < 0 \) (resp. \( g^2(y) - 1 > 0 \)). Therefore, we have:

**Proposition 3.1** A translation surface of Type 1 in \( G^1_3 \) is spacelike (resp. timelike) if and only if its generating isotropic curve is spacelike (resp. timelike).

The Gaussian curvature of a surface (4) is given by
\[ K = -\epsilon \frac{f''(x)g''(y)}{(1 - g'^2(y))^2}, \]
and the mean curvature for a spacelike surface by
\[ H = \frac{g''(y)}{2(1 - g'^2(y))^{3/2}}, \]
and for a timelike surface by
\[ H = \frac{g''(y)}{2(g^2(y) - 1)^{3/2}}. \]

By ‘’ we have denoted derivatives with respect to corresponding variables.

First we examine surfaces of Type 1 having constant Gaussian curvature. Contrary to the Euclidean case, since variables \( x, y \) in the function \( K \) can be separated, \( K \) is constant if and only if either
\[ f''(x) = \text{const.} \neq 0 \quad \text{and} \quad \frac{g''(y)}{(1 - g'^2(y))^2} = \text{const.} \neq 0, \]  
\[ f''(x) = 0 \quad \text{or} \quad \frac{g''(y)}{(1 - g'^2(y))^2} = 0. \]
Therefore, by solving (5) we obtain \( f(x) = ax^2 + bx + c \), \( a, b, c \in \mathbb{R} \) with 

\[
g(y) = \frac{g'(y)}{1 - g'^2(y)} + \frac{1}{2} \log \frac{1 + g'(y)}{1 - g'(y)} = Ay + B, \quad A, B \in \mathbb{R}.
\]

The previous ordinary differential equation can not be solved in elementary functions. Therefore, we will reparametrize the generating curve \( \beta(u) = (0, h(u), k(u)) \), \( h'^2(u) - k'^2(u) = \pm 1 \). For this parametrization we get

\[
\sigma(u, v) = (0, h'(u), k'(u)), \quad N(u, v) = (0, k'(u), h'(u)), \quad W(u, v) = 1.
\]

Therefore

\[
L = -\epsilon f''(x)h'(u), \quad M = 0, \quad N = \epsilon(k'(u)h''(u) - h'(u)k''(u)).
\]

Since \( h'^2(u) - k'^2(u) = \pm 1 \), then \( h'(u)h''(u) - k'(u)k''(u) = 0 \), and therefore

\[
N = \frac{k''(u)}{f'(u)}. \quad \text{Now we get} \quad K = f''(x)k''(u), \quad \text{regardless whether a surface is spacelike or timelike. By solving the equation} \quad K = \text{const.}, \quad \text{we get} \quad k(u) = \frac{1}{2}Au^2 + Bu + C \quad \text{and}
\]

\[
h(u) = \frac{Au + B}{2A} \sqrt{1 + (Au + B)^2} + \frac{1}{2A} \arcsinh(Au + B) + C_1, \quad A, B, C, C_1 \in \mathbb{R}, \quad (7)
\]

in the case when a surface (i.e. generating curve \( \beta \) is spacelike) and

\[
h(u) = \frac{Au + B}{2A} \sqrt{(Au + B)^2 - 1} + \frac{1}{2A} \log \left( (Au + B) + \sqrt{(Au + B)^2 - 1} \right) + C_1, \quad (8)
\]

in the case when a surface timelike (i.e. generating curve \( \beta \) is timelike), see Figures 3, 5.

We can notice that parametrizations with (7), (8) are parametrizations with principal curves of the surface.

Furthermore, from (6) we have that the Gaussian curvature is constant the obtained surface is cylindrical

\[
z(x, y) = ax + b + g(y), \quad \text{or} \quad z(x, y) = f(x) + cy + d.
\]

Therefore, we have proved in the pseudo-Galilean space the following theorem:

**Theorem 3.2** If \( S \) is a translation surface of Type 1 of constant Gaussian curvature in the pseudo-Galilean space, then \( S \) is congruent to a special surface with \( f(x) = ax^2 + bx + c \), \( a, b, c \in \mathbb{R} \) and \( k, h \) given by (7) for a spacelike surface or (8) for a timelike surface \( (K \neq 0) \), or to a cylindrical surface \( (K = 0) \) having either non-isotropic or isotropic rulings.
Now we examine surfaces of Type 1 of constant mean curvature.

**Theorem 3.3** If $S$ is a translation surface of Type 1 of constant mean curvatur
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Figure 5: Generating curves for timelike surfaces with \( K \neq 0 \) – a parabola and the isotropic curve (8)

**curvature** \( H \neq 0 \) in the pseudo-Galilean space, then \( S \) is congruent to a surface

\[
z = f(x) + \frac{1}{2H} \sqrt{(2Hy + c_1)^2} \pm 1 + c_2, \quad c_1, c_2 = \text{const.} \tag{9}
\]

Specially, if \( f(x) = ax + b, a, b, \in \mathbb{R} \), then \( S \) is congruent to a hyperbolic sphere (an equilateral hyperbolic cylinder).

**Proof.** If we put \( h(y) = g'(y) \), one should solve the ordinary differential equation

\[
h'(y) = 2H(1 - h^2(y))^{\frac{3}{2}}, \quad H = \text{const}
\]

in the case when a surface is spacelike, and

\[
h'(y) = 2H(h^2(y) - 1)^{\frac{3}{2}}, \quad H = \text{const}
\]

in the case when a surface is timelike. We get

\[
h(y) = \pm \frac{2Hy + c_1}{\sqrt{(2Hy + c_1)^2} + 1} \tag{10}
\]

and

\[
h(y) = \pm \frac{2Hy + c_1}{\sqrt{(2Hy + c_1)^2} - 1}, \quad c_1 \in \mathbb{R}. \tag{11}
\]

Furthermore, notice that curves \( \beta(y) = (0, y, h(y)) \) satisfy the equation \( y^2 - z^2 = \text{const.} \) and therefore they are hyperbolas. If the curve \( \alpha \) is a line, then \( S \) is a hyperbolic sphere (an equilateral hyperbolic cylinder), see Figure 1. A unit sphere \( y^2 - z^2 = -1 \) (resp. \( y^2 - z^2 = 1 \)) is a spacelike (resp. timelike) surface.

Notice that contrary to the Euclidean situation, a surface having constant mean curvature need not be ruled, due to the fact that \( H \) is a function of a variable \( y \) only.
Furthermore, notice that among surfaces with $K = \text{const.}$ or $H = \text{const.}$ there is also a parabolic sphere (a parabolic cylinder). A unit parabolic spacelike (resp. timelike) sphere in a pseudo-Galilean space is a surfaces given by (its normal form) $z = \frac{1}{2}x^2$ (resp. $y = \frac{1}{2}x^2$). It is an admissible surface, a cylindrical surface with isotropic rulings, having $K = H = 0$, see Figure 6.

Finally, for minimal translation surfaces in $G_3^1$ the following statement is obvious.

**Theorem 3.4** If $S$ is a translation surface of Type 1 of zero mean curvature in the pseudo-Galilean space, then $S$ is congruent to a cylindrical surface with isotropic rulings (and therefore $K = 0$)

$$z = f(x) + ay + b, \ a, b \in \mathbb{R}.$$ 

In other words, the obtained surface is a ruled surface with rulings having the constant isotropic direction $(0, 1, a)$. It is spacelike (resp. timelike) if and only if $1 - a^2 > 0$ (resp. $1 - a^2 < 0$). The obtained results agree with the statement of Theorem 2.1.

## 4 Translation Weingarten surfaces

Weingarten surfaces are surfaces whose Gaussian and mean curvature satisfy a functional relationship (of class $C^0$ at least). The class of Weingarten surfaces contains already mentioned surfaces of constant curvatures $K$ or $H$. Furthermore, a $C^r$-surface, $r > 3$, is Weingarten if and only if $K_x H_y - K_y H_x = 0$.

In the case of translation surfaces of Type 1 the following theorem holds:

**Theorem 4.1** A translation surface of Type 1 in the pseudo-Galilean space is a Weingarten surface if and only if it is either (a part of)
1. an isotropic plane,
2. a cylindrical surface with isotropic or non-isotropic rulings,
3. a translation surface of constant Gaussian curvature of Theorem 3.2,
4. a translation surface of constant mean curvature of Theorem 3.3,
5. a surface \( z = ax^2 + bx + c + g(y) \), \( a, b, c \in \mathbb{R} \).

**Proof.** Since \( H \) is a function of \( y \)-variable only, a \( C^r \)-surface, \( r > 3 \), is Weingarten if and only if \( K_x H_y = 0 \). This condition is satisfied either if \( K_x = 0 \) or \( H_y = 0 \). The latter condition gives surfaces of constant mean curvature (Theorem 3.3, Theorem 3.4). The first condition

\[
K_x = \frac{f'''(x)g''(y)}{(1 - g'^2(y))^4}
\]

is satisfied by surfaces of constant Gaussian curvature (Theorem 3.2) or surfaces which have \( f'''(x) = 0 \). These are the surfaces

\[
z = ax^2 + bx + c + g(y), \quad a, b, c \in \mathbb{R}.
\]

Among them, depending on a curve \( \beta \), there are spacelike \( 1 - g'^2(y) > 0 \) as well as timelike \( 1 - g'^2(y) < 0 \) surfaces. Notice, that the functional relationship between \( K \) and \( H \) of these surfaces is given by

\[
(1 - g'^2(y))^{1/2}K - 2H = 0.
\]

\[\square\]

![Figure 7: Translation Weingarten surface of type 1](image)

In Euclidean space, if the curvatures \( K, H \) of a surface satisfy \( aK + bH = 0 \), with at least one of the constant \( a, b \) different from 0, then a surface is the Scherk’s minimal surface or a surface is flat ([11]).
Proposition 4.2  The only translation Weingarten surfaces of Type 1 in the pseudo-Galilean space that satisfy linear condition $aK + bH = 0$, with at least one of the constant $a, b$ different from 0, are cylindrical surfaces with non-isotropic or isotropic rulings (specially parabolic spheres).

Proof. If $a = 0, b \neq 0$, then $S$ is a minimal surface of Theorem 3.4. If $a \neq 0, b = 0$ then $S$ is flat surface of Theorem 3.2. If $ab \neq 0$, then the condition $aK + bH = 0$ implies either $g''(y) = 0$ or $f''(x) = W(y) = \text{const.} = A$. Therefore we get either $g(y) = ay + b$, which generates cylindrical surfaces with isotropic rulings, or $f(x) = \frac{A}{2}x^2 + Bx + C$ and $g(y) = ay + b$, $A, B, C, a, b \in \mathbb{R}$. These surfaces are parabolic spheres.

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