On Near Left Almost Rings

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Abstract. In this paper we give the notion of near left almost ring (abbreviated as nLA-ring) \((R, +, \cdot)\), i.e. \((R, +)\) is an LA-group, \((R, \cdot)\) is an LA-semigroup and one distributive property of ‘·’ over ‘+’ holds, where both the binary operations “+” and “·” are non-associative. An nLA-ring is a generalization of an LA-ring and footed parallel to the near ring.

Mathematics Subject Classification: 16A76, 20M25, 20N02

Keywords: Near ring, LA-ring, Near LA-ring

1. Introduction and Preliminaries

The concept of a left almost semigroup (abbreviated as an LA-semigroup) was introduced by M. Kazim and M. Naseeruddin [1], which is in fact a generalization of commutative semigroup. A groupoid \((S, \ast)\) is called an LA-semigroup if, \((a \ast b) \ast c = (c \ast b) \ast a\) for all \(a, b, c \in S\), holds. It is also known as an Abel-Grassmann’s groupoid (abbreviated as an AG-groupoid), for instance see [6]. Later, the structure was explored in [2] and [4]. Further in [3], the concept is extended to the left almost group LA-group (i.e., a non-empty set \(G\) with a binary operation “∗” such that \((G, \ast)\) is an LA-semigroup having left identity \(e\) and each element of \(G\) has left inverse). LA-group is a non associative structure but has a sort of resemblance with a commutative group.

By [8], a non-empty set \(R\) with two binary operations “+” and “·” is called a left almost ring (LA-ring) if \((R, +)\) is an LA-group, \((R, \cdot)\) is an LA-semigroup and distributive laws of “·” over “+” hold. This structure enhanced [7] as a generalization of commutative semigroup rings.

By [5] a near-ring is a non-empty set \(N\) together with two binary operations “+” and “·” such that \((N, +)\) is a group (not necessarily abelian), \((N, \cdot)\) is a semigroup and one sided distributive (left or right) of “·” over “+” holds. The theory of near-ring runs completely parallel in both cases (left near-ring or right near-ring). In this study, we consider left near-ring. Most examples of near-rings show, \(0n = 0\) and \((-n)m = -nm\) do not hold in general. One
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therefore defines for a near-ring $N$: (i). $N_0 = \{n \in N : 0n = 0\}$ is called a zero-symmetric part of $N$. (ii). $N_c = \{n \in N : 0n = n\} = \{m \in N : \forall n \in N : nm = m\}$ is called a constant part of $N$.

In this study, we introduce the notion of a near left almost ring (abbreviated as an nLA-ring), which is in fact a generalization of a left almost ring. It possesses properties which we usually encounter in "near ring" and "LA-ring". We observe that properties which owned by an LA-ring are mostly true for an nLA-ring. Also we see that in an nLA-ring the zero symmetric part and the constant part do not exist while they do in near-ring. Although, this structure is non-commutative and non-associative but due to its structural properties it behaves like a commutative ring and a commutative near ring.

2. Near LA-Ring

In this section, we define a near left almost ring and give few examples. Also we touch its elementary properties and discuss the shape of substructures.

2.1. Definition and examples.

**Definition 1.** A non empty set $N$ with two binary operation "+" and "·" is called a near left almost ring (or simply an nLA-ring) if and only if

(nLA1). $(N, +)$ is an LA-group.

(nLA2). $(N, \cdot)$ is an LA-semigroup.

(nLA3). Left distributive property of $\cdot$ over $+$ holds.

i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in N$.

The near LA-ring (nLA-ring) is denoted in ordered triplet as $(N, +, \cdot)$.

In view of (nLA3), one speaks more precisely of "left near LA-ring". Postulating (nLA3a) $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in N$.

Instead of (nLA3), one gets "right near LA-ring". The theory runs completely parallel in both cases.

In this paper we through out consider left near LA-ring.

**Example 1.** $N = \{a, b, c, d, e\}$ is an nLA-ring with binary operations "∗" and "⊙" defined on it which are as follows;

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**Example 2.** Let $(N, +, \cdot)$ be an nLA-ring, then $A = \{f : f : N \longrightarrow N\}$ is an nLA-ring with binary operations defined as follows

$(f + g)(n) = f(n) + g(n)$ and $(f \cdot g)(n) = f(n) \cdot g(n)$ for all $n \in N$. 

Remark 1. An LA-ring is an nLA-ring but a near-ring does not implies nLA-ring.

Proposition 1. Let \((F,+,\cdot)\) be any field. Then \((F,\ast,\odot)\) is an nLA-ring by defining the binary operations as: for \(a,b,c \in F\), \(a \ast b = b - a\) and 
\[ a \odot b = \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0 \\ b \cdot a^{-1} & \text{other wise} \end{cases}. \]

Proof. Let \(F\) be a field, 0 and 1 are additive and multiplicative identities respectively. By \([3]\), \((F,\ast)\) is an LA-group. By theorem \([4,\text{ theorem 2.1}]\), \((F \setminus \{0\},\odot)\) is an LA-semigroup. Let \(a,b,c \in F\), if any of \(a,b\) and \(c\) is zero, then 
\((a \odot b) \odot c = (c \odot b) \odot a = 0\), so \((F,\odot)\) is an LA-semigroup. Now for distributive law, let for each \(a,b,c \in F\), then \(a \odot (b \ast c) = a \odot (c - b) = (c - b) \cdot a^{-1} = c \cdot a^{-1} - b \cdot a^{-1} = a \odot c - a \odot b = a \odot b \ast a \odot c\). Consider \((b \ast c) \odot a = (c - b) \odot a = a \cdot (c - b)^{-1}\). Since in general \((c - b)^{-1} \neq c^{-1} - b^{-1}\), so \((b \ast c) \odot a \neq b \odot a \ast a \odot c\). It means that right distributive law of \(\odot\) over \(\ast\) does not hold. Hence \((F,\ast,\odot)\) is an nLA-ring.

2.2. Elementary properties of an nLA-Ring. We give few basic properties of an nLA-ring, some of which are different than that of a near-ring.

Theorem 1. If \((N,+,\cdot)\) is an nLA-ring with additive left identity 0, then for all \(a,b \in N\) we have.
1. \(a0 = 0\).
2. \(0a = 0\).
3. \(a(-b) = -ab\).
4. \((-a) = a\).
5. \((-a + b) = -a - b\).

Remark 2. It is important to note that in a near-ring (2) does not hold in general. It means that in an nLA-ring both the zero-symmetric part and the constant part do not exist.

Definition 2. An nLA-ring \((N,+,\cdot)\) with left identity 1, such that \(1 \cdot b = b\) for all \(b \in N\), is called an nLA-ring with left identity.

A non trivial right near-ring with identity exists but in case of a right nLA-ring with left identity, the following proposition speaks differently.

Proposition 2. A right nLA-ring \(N\) with left identity 1, is an LA-ring.
Proof. For \(a,b,c \in N\)
\[ a \cdot (b + c) = (1 \cdot a) \cdot (b + c) = ((b + c) \cdot a) \cdot 1, \ 	ext{by left invertive law} \]
\[ = (b \cdot a + c \cdot a) \cdot 1, \ 	ext{by right distributive} \]
\[ = (b \cdot a) \cdot 1 + (c \cdot a) \cdot 1, \ 	ext{by right distributive} \]
\[ = (1 \cdot a) \cdot b + (1 \cdot a) \cdot c, \ 	ext{by left invertive law} \]
\[ = a \cdot b + a \cdot c. \]
It means that a right near LA-ring with left identity implies LA-ring.

**Remark 3.** A right nLA-ring \( N \) with left identity 1 does not exist.

**Definition 3.** An element \( a \) in an nLA-ring \( (N, +, \cdot) \) is called a left zero if \( a \cdot b = a \) similarly, \( a \) is a right zero if \( b \cdot a = a \), and if \( a \) is both left and right zero, then \( a \) is called a zero element of an nLA-ring \( (N, +, \cdot) \).

**Example 3.** In example 1, \( a \) is a zero element.

**Definition 4.** An element \( d \) of an nLA-ring \( N \) is called distributive if for all \( n, m \in N \) such that \( (n + m) \cdot d = nd + md \). The set of all distributive elements of an nLA-ring \( N \) is denoted by \( N_d = \{d \in N : d \) is distributive\}.

### 2.3. Substructures of an nLA-ring

We begin with the following definition.

**Definition 5.** A non empty subset \( S \) of an nLA-ring \( N \) is said to be an nLA-subring if and only if \( S \) is itself an nLA-ring under the same binary operations as in \( N \).

**Example 4.** Let \( N = \{a, b, c, d, e, f\} \) be an nLA-ring under the binary operations defined as follows,

\[
\begin{array}{cccccc}
* & a & b & c & d & e & f \\
\hline
a & a & b & c & d & e & f \\
b & f & a & b & c & d & e \\
c & e & f & a & b & c & d \\
d & d & e & f & a & b & c \\
e & c & d & e & f & a & b \\
f & f & c & d & e & f & a \\
\end{array}
\quad
\begin{array}{cccccc}
\odot & a & b & c & d & e & f \\
\hline
a & a & a & a & a & a & a \\
b & b & a & b & c & d & e \\
c & c & a & c & e & a & c \\
d & d & a & d & a & d & a \\
e & e & c & a & e & c & e \\
f & f & e & d & c & b & a \\
\end{array}
\]

Let \( S = \{a, c, e\} \) such that

\[
\begin{array}{cccc}
* & a & c & e \\
\hline
a & a & c & e \\
c & c & a & e \\
e & e & c & a \\
\end{array}
\quad
\begin{array}{cccc}
\odot & a & c & e \\
\hline
a & a & a & a \\
c & a & e & c \\
e & a & c & e \\
\end{array}
\]

Here \( S \) is an nLA-subring of an nLA-ring \( N \).

**Theorem 2.** A non-empty subset \( S \) of an nLA-ring \( (N, +, \cdot) \) is an nLA-subring if and only if \( a - b \in S \) and \( a \cdot b \in S \) for all \( a, b \in S \).

**Proposition 3.** Intersection of two nLA-subrings of an nLA-ring is an nLA-subring.

**Corollary 1.** Intersection of any number of nLA-subrings of an nLA-ring is an nLA-subring.

**Proposition 4.** Let \( N \) be an nLA-ring with left identity 1, then

\[
N_d = \{d \in N : d \text{ is distributive}\} \quad \text{is an nLA-subring of an nLA-ring} \ N.
\]
Definition 6. An nLA-subring \( I \) of an nLA-ring \( N \) is called a left ideal of \( N \) if \( NI \subseteq I \), and \( I \) is called a right ideal if for all \( n, m \in N \) and \( i \in I \) such that \((i+n)m-nm \in I\), and is called two sided ideal or simply ideal if it is both left and right ideal.

Example 5. In example 4, the nLA-subring \( S \) is an ideal of \( N \).

Remark 4. In an nLA-ring \((N, +, \cdot)\), \(\{0\} \) and \(N\) are ideals of \(N\) called improper ideals of \(N\).

Corollary 2. Intersection of any number of ideals of an nLA-ring is an ideal.

3. Factor Near LA-ring

Let \( I \) be an ideal of an nLA-ring \( N \). Then “\(\equiv\)” is an equivalence relation on \( N \) defined by \( a \equiv b \ (\text{mod} \ I) \) if and only if \( a-b \in I \). This equivalence relation partitions \( N \) into equivalence classes. The set of all equivalence classes is denoted by \( N/I \), i.e. \( N/I = \{[n] = I + n : n \in N\} \). Now we define binary operations on \( N/I \) as follows:

\[(I+n) + (I+m) = I + (n+m) \quad \text{and} \quad (I+n)(I+m) = I + nm, \text{ where } I+n, I+m \in N/I.\]

These binary operations are well-defined. Indeed; suppose that \( I+n = I+m \) and \( I+x = I+y \) implies that \( n \in I+m \) and \( x \in I+y \). That is \( n = i + m \) and \( x = j + y \) for some \( i, j \in I \). Now consider

\[n+x = (i+m) + (j+y) = (i+j) + (m+y) \in I+(m+y) \implies I+(n+x) = I+(m+y).\]

Similarly

\[nx = (i+m)(j+y) = (i+m)j + (i+m)y\]
\[= (0 + (i+m)j) + (i+m)y = ((i+m)y + (i+m)j) + 0\]
\[= ((i+m)y + (i+m)j) + (-my + my)\]
\[= ((0 + (i+m)y) + (i+m)j) + (-my + my)\]
\[= (((i+m)j + (i+m)y) + 0) + (-my + my)\]
\[= (((i+m)j + (i+m)y) - my) + (0 + my)\]
\[= ((-my + (i+m)y) + (i+m)j) + my\]
\[= (((0 - my) + (i+m)y) + (i+m)j) + my\]
\[= (((i+m)j - my) + 0) + (i+m)j + my \in I+my.\]

Hence \( nx \in I+my \). This implies that \( I+nx = I+my \)

These binary operations in \( N/I \) constitute the following proposition.

Proposition 5. \( N/I \) is an nLA-ring.

We call \( N/I \), a factor (or a quotient) nLA-ring.
4. Near LA-ring Homomorphism

In this section, we define an nLA-ring homomorphism and establish the fundamental theorems of homomorphism for an nLA-ring.

4.1. Step up.

**Definition 7.** Let $N$ and $\hat{N}$ be two nLA-rings. A map $\varphi : N \rightarrow \hat{N}$ is called a near LA-ring homomorphism if for all $n, m \in N$ such that

$$\varphi (n + m) = \varphi (n) + \varphi (m) \quad \text{and} \quad \varphi (nm) = \varphi (n) \varphi (m) .$$

One may call it an nLA-ring homomorphism.

(a) An nLA-ring homomorphism is a monomorphism if it is one-one. 
(b) An nLA-ring homomorphism is an epimorphism if it is onto.  
(c) An nLA-ring homomorphism is a isomorphism if it is both one-one and onto.

**Remark 5.** An nLA-ring homomorphism is in fact, LA-group homomorphism and LA-semigroup homomorphism respectively.

**Theorem 3.** If $\varphi : N \rightarrow N'$ is an nLA-ring homomorphism, then

(1) $\varphi (0_N) = 0_{N'}$.  
(2) $\varphi (-n) = -\varphi (n)$.  
(3) $\varphi (n - m) = \varphi (n) - \varphi (m)$.

**Definition 8.** If $\varphi : N \rightarrow N'$ is an nLA-ring homomorphism, then kernal of $\varphi$ is defined by $\ker \varphi = \varphi^{-1} \{0_N\}$.

**Proposition 6.** Let $\varphi : N \rightarrow N'$ be an nLA-ring epimorphism and $I$ be an ideal of $N$, then $\varphi (I)$ is an ideal of $N'$.

**Proposition 7.** Let $\varphi : N \rightarrow N'$ be an nLA-ring homomorphism then,

**Lemma 1.** (1) if $K$ is an ideal of $N'$ then, $\varphi^{-1} (K)$ is an ideal of $N$, containing $\ker \varphi$.  
(2) $\ker \varphi$ is an ideal of $N$.

**Proposition 8.** If $\varphi : N \rightarrow N'$ is an nLA-ring homomorphism, then $\ker \varphi = \{0_N\}$ if and only if $\varphi$ is an nLA-ring monomorphism.

4.2. Isomorphism Theorems. On the discussion of previous section we state the following theorem.

**Theorem 4.** Let $\varphi : N \rightarrow N'$ be an nLA-ring epimorphism from nLA-ring $N$ to $N'$. Then $N/\ker \varphi \cong N'$. In general if $\varphi$ is a homomorphism, then $N/\ker \varphi \cong \text{Im} \varphi$.

We call theorem 4, the first isomorphism theorem for an nLA-ring.

**Remark 6.** $\pi : N \rightarrow N/I$ is the canonical nLA-ring epimorphism.
Theorem 5. If $I$ and $J$ are two ideals of an nLA-ring $N$, then $(I + J)/I \cong J/(I \cap J)$.

Proof. Since $I = I + 0 \subseteq I + J$ and $I \cap J \subseteq J$ therefore, $(I + J)/I$ and $J/(I \cap J)$ are well-defined. Now we define a map $\varphi : (I + J) \rightarrow J/A$ by $\varphi (i + j) = A + j$ for all $i + j \in I + J$, where $A = I \cap J$.

Let $i + j = i' + j'$, by applying the left invertive law and the medial law we have:

\[ i - i' = -j + j' \in A \]

Trivially it is onto. For an nLA-ring homomorphism, let $i + j, i' + j' \in I + J$

\[ \varphi ((i + j) + (i' + j')) = \varphi ((i + i') + (j + j')) = A + (j + j') = (A + j) + (A + j') = \varphi (i + j) + \varphi (i' + j'). \]

Similarly $\varphi ((i + j)(i' + j')) = \varphi [((i + j)j' - jj') + 0] + (i + j)i' + jj' = A + jj' = (A + j)(A + j') = \varphi (i + j) \varphi (i' + j').$

By the 1st isomorphism theorem we have $(I + J)/\ker \varphi \cong J/A$. Now let $i + j \in \ker \varphi$, where $i \in I$ and $j \in J$, then $\varphi (i + j) = A$ by definition of $\varphi$, $A + j = A$ implies that $j \in A = I \cap J$ and hence $\ker \varphi \subseteq I$.

Conversely, let $i \in I$, then $i = (i + 0) + 0 \in I + J$, so $\varphi (i) = \varphi ((i + 0) + 0) = A + 0 = A$ this implies that $i \in \ker \varphi$, and hence $I \subseteq \ker \varphi$. Therefore $I = \ker \varphi$, hence $(I + J)/I \cong J/(I \cap J)$. ■

We call theorem 5, the 2nd Isomorphism theorem for an nLA-ring.

Theorem 6. Let $I$ and $J$ be ideals of an nLA-ring $N$ such that $I \subseteq J$, then $N/J \cong (N/I)/(J/I)$.

We call theorem 6, the 3rd Isomorphism theorem for an nLA-ring.

We call theorems 4, 5 and 6 the fundamental theorems of nLA–ring homomorphism.

5. A Near integral domain and a Near almost field

In this section we extend this structure by defining unit elements, zero divisors, near integral domain, near almost field and discuss some results regarding generalization of the LA-rings and parallel to the near-rings.

Definition 9. Let $(N, +, \cdot)$ be an nLA-ring with left identity 1. An element $0 \neq n \in N$ is called unit if there exists $m \in N$ such that $n \cdot m = m \cdot n = 1$. The set of all unit elements of $N$ is denoted by $U(N)$.

Theorem 7. The set of all unit elements of an nLA-ring $(N, +, \cdot)$ with left identity 1, forms an LA-group under “·”.
Proof. Let \((N, +, \cdot)\) be an nLA-ring with left identity 1, and \(U(N)\) be the set of all unit elements of \(N\). Since \(1 \cdot 1 = 1\) this implies that \(1 \in U(N)\), so \(U(N) \neq \emptyset\). Let \(a, b \in U(N)\), then there exist \(a', b' \in N\) such that \(a \cdot a' = a' \cdot a = 1\) and \(b \cdot b' = b' \cdot b = 1\).

Now consider \((a \cdot b) \cdot (a' \cdot b') = (a \cdot a') \cdot (b \cdot b') = 1 \cdot 1 = 1\) (by using medial law).

Similarly, \((a' \cdot b') \cdot (a \cdot b) = (a' \cdot a) \cdot (b' \cdot b) = 1 \cdot 1 = 1\) (by using medial law).

This shows that \(a' \cdot b'\) is the inverse of \(a \cdot b\), hence \(a \cdot b \in U(N)\). It means that \(U(N)\) is closed under “\(\cdot\).” Left invertive law holds in \(U(N)\) as it holds in \(N\), therefore \(U(N)\) is an LA-monoid with left identity 1. For any \(a \in U(N)\) there exists \(a' \in S\) such that \(a \cdot a' = a' \cdot a = e\) implies that \(a' \in U(N)\). Hence \(U(N)\) is an LA-group.

**Definition 10.** Let \((N, +, \cdot)\) be an nLA-ring.

(a) An element \(a \in N\) is called left (right) cancellative if \(a \cdot b = a \cdot c\), then \(b = c\) (\(b \cdot a = c \cdot a\) then \(b = c\)), where \(a, b, c \in N\), and \(a\) is called cancellative if it is both left and right cancellative. However the nLA-ring \(N\) is called cancellative if each element in \(N\) is cancellative.

(b) A non-zero element \(a\) of \(N\) is called a left zero divisor if there exists \(0 \neq b \in N\) such that \(a \cdot b = 0\). Similarly \(a\) is a right zero divisor if \(b \cdot a = 0\). If \(a\) is both a left and a right zero divisor, then \(a\) is called a zero divisor.

**Example 6.** Let \(N = \{a, b, c, d\}\) be a set with two binary operation “*” and “*” defined on it which are as follows

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This shows that \(c\) is a zero divisor in \(N\), whereas \(a\) is a zero element in \(N\).

**Theorem 8.** An nLA-ring \((N, +, \cdot)\) has no left zero divisor if and only if left cancellative law holds.

**Definition 11.** An nLA-ring \((N, +, \cdot)\) with left identity 1, is called an nLA-domain if it has no left zero divisor.

**Example 7.** \((N, *, \odot)\) of example 1, is an nLA-domain.

**Definition 12.** An nLA-ring \((F, +, \cdot)\) with left identity 1 is called a near field (n-almost field) if and only if each non-zero element of \(F\) has inverse under “\(\cdot\).”

**Theorem 9.** Every finite nLA-domain is an n-almost field.

**Proof.** Let \(D = \{1, x_1, x_2, x_3, \ldots, x_{n-1}\}\) be a finite nLA-domain under the binary operations “\(\cdot\)” and “\(*\)” with left identity 1. Let \(0 \neq x \in D\), then we have \(D_x = \{x, xx_1, xx_2, xx_3, \ldots, xx_{n-1}\}\). Clearly \(D_x \subseteq D\), Now let \(xx_i = xx_j\), where \(i \neq j\), then \(x(x_i - x_j) = 0\).
Since \( x \neq 0 \) and \( D \) is an nLA-integral domain, then \( x_i - x_j = 0 \). So this implies that \( x_i = x_j \). Which is not possible because \( |D| = n \). So \( D_x \) is a permutation of \( D \). That is, \( |D_x| = n \) hence \( D_x = D \). Since \( 1 \in D = D_x \) therefore, \( 1 = xx_i \). This shows that \( x_i \) is multiplicative right inverse of \( x \) since \( D \) is an nLA-ring with left identity \( 1 \), so it is multiplicative left inverse of \( x \). Therefore, each non-zero element of \( D \) has left inverse in \( D \). Hence \( D \) is an n-almost field.

**Example 8.** Let \( D = \{a, b, c, d, e, f, g\} \) be a set with two binary operation “\( * \)” and “\( \circ \)” defined on it which are as follows

\[
\begin{array}{cccccc}
| & a & b & c & d & e & f & g \\
\hline
a & a & b & c & d & e & f & g \\
b & g & a & b & c & d & e & f \\
c & f & g & a & b & c & d & e \\
d & e & f & g & a & b & c & d \\
e & d & e & f & g & a & b & c \\
f & c & d & e & f & g & a & b \\
g & b & c & d & e & f & g & a \\
\end{array}
\]

and

\[
\begin{array}{cccccc}
| & a & b & c & d & e & f & g \\
\hline
a & a & a & a & a & a & a & a \\
b & g & a & b & c & d & e & f \\
c & f & g & a & b & c & d & e \\
d & e & f & g & a & b & c & d \\
e & d & e & f & g & a & b & c \\
f & c & d & e & f & g & a & b \\
g & b & c & d & e & f & g & a \\
\end{array}
\]

Hence \( (D, *, \circ) \) is a finite nLA-integral domain and hence it is an n-almost field.

**Proposition 9.** An n-almost field has no proper ideal.

**References**


Received: April, 2010