\( \pi g\alpha \)-Separation Axioms
in BiČech Spaces

Ganes M. Pandya
School of Petroleum Management, Gujarat, India
ganes_17@yahoo.com

C. Janaki
Sree Narayana Guru College, Coimbatore, India

I. Arockiarani
Nirmala College for Women, Coimbatore, India

Abstract

In this paper, we introduce the concepts of \( \pi g\alpha \)-generalized closed sets in biČech closure space and investigate some of its Properties and characterization.

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1 Introduction

Čech closure spaces were introduced by Čech [1]. In Čech’s approach the operator satisfies idempotent condition among Kuratowski axioms. This condition need not hold for every set \( A \) of \( X \). When this condition is also true, the operator becomes topological closure operator. Thus the concept of closure space is the generalisation of a topological space. Closure functions that are more general than the topological ones have been studied already by Day[3]. A thorough discussion on closure functions is due to Hammer, see eg [6,7] and more recently Gnilka[4]. The notion of bitopological space were introduced by
J.C. Kelly[8]. Such spaces are equipped with two arbitrary topologies. Furthermore, Kelly extended some of the standard results of separation axioms in a topological space to a bitopological space. Čech closure space were studied by Chandrasekharra Rao in [2]. In this paper we introduce the $\pi g \alpha$-closed sets in BiČech closure spaces.

2 Preliminaries

**Definition 2.1.** Two functions $k_1$ and $k_2$ from power set $X$ to itself are called Čech closure operators (simply biclosure operator) for $X$ if they satisfies the following properties

- $k_1(\emptyset) = \emptyset$ and $k_2(\emptyset) = \emptyset$.
- $A \subset k_1(A)$ and $A \subset k_2(A)$ for any set $A \subset X$.
- $k_1(A \cup B) = k_1(A) \cup k_2(B)$ and $k_2(A \cup B) = k_2(A) \cup k_2(B)$ for any $A, B \subset X$.

$(X, k_1, k_2)$ is called biČech closure space.

A subset $A$ is closed in the closure space $(X, k)$ if $k(A) = A$ and it is open if its complement is closed. The empty set and the whole space are both open and closed. $A$ is a closed subset of a biČech closure space $(X, k_1, k_2)$ if and only if $A$ is a closed set of $(X, k_1)$ and $(X, k_2)$. Also the following conditions are equivalent for a closed set $A$

$$k_2k_1(A) = A$$
$$k_1A = A \ , \ k_2A = A$$

**Definition 2.2.** [2] A subset $A$ in a biČech closure space $(X, k_1, k_2)$ is said to be

1. $k_i$-regular open if $A = \text{int}_{k_i}(k_i(A))$, $i = 1, 2$
2. $k_i\alpha$-closed if $k_i[\text{int}_{k_i}(k_i(A))] \subseteq A$, $i=1,2$

The smallest Čech $\alpha$-closed set containing $A$ is called Čech $\alpha$-closure of $A$ and it is denoted by $k_\alpha(A)$. The largest Čech $\alpha$-open set contained in $A$ is called Čech $\alpha$-interior of $A$ and is denoted by $\text{int}_{k_\alpha}(A)$. $k_1\alpha(A)$ and $k_2\alpha(A)$ is the intersection of all Čech $\alpha$-closed sets under the operator $k_1$ and $k_2$. The finite union of Čech regular-open sets is called Čech $\pi$-open. Every Čech regular open set is Čech $\pi$-open but the converse need not be true.

3. $(k_1, k_2) - \pi g\alpha$ closed sets

**Definition 3.1.** A subset $A$ in a Čech closure space $(X, k_1, k_2)$ is said to be $(k_1, k_2)$-$(\pi g\alpha)$ closed if $k_2\alpha(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $k_1\pi$ open set in $X$.

**Example 3.2.** Let $X = \{a, b, c\}$ and let $k_1$ and $k_2$ be the two operators defined as $k_1(\phi) = \phi$, $k_1(X) = k_1(\{a, b\}) = X$, $k_1(\{a\}) = \{a, c\}$, $k_1(\{b\}) = k_1(\{b, c\}) = \{b\}$, $k_1(\{c\}) = \{c\}$.

$k_2(\phi) = \phi$, $k_2(X) = k_2(\{b, c\}) = X$, $k_2(\{a\}) = \{a\}$, $k_2(\{b\}) = k_2(\{a, b\}) = \{a, b\}$, $k_2(\{c\}) = k_2(\{a, c\}) = \{a\}$.

In this set $\{b\} \subseteq \{a, b\}$ is a $(k_1, k_2) - \pi g\alpha$ closed set.

**Proposition 3.3.** If $A$ and $B$ are $(k_1, k_2) - \pi g\alpha$ closed sets then so is $A \cup B$.

**Proof.** Let $A$ and $B$ be two $(k_1, k_2) - \pi g\alpha$ closed sets. Let $U$ be $k_1\pi$ open set in $X$. Let $(A \cup B) \subseteq U$. Since $A$ and $B$ are $(k_1, k_2) - \pi g\alpha$-closed sets, $K_{2\alpha}(A) \subseteq U$ and $k_{2\alpha}(B) \subseteq U$. Hence $k_{2\alpha}(A \cup B) \subseteq U$. Thus $A \cup B$ is $(k_1, k_2) - \pi g\alpha$ closed set.
Proposition 3.4. If $A$ is $(k_1, k_2)$-$\pi g\alpha$ closed set then $k_{2\alpha}(A) - A$ contains no non-empty $k_1$-$\pi$ closed sets.

Proof. Let $A$ be $((k_1, k_2)$-$\pi g\alpha$ closed set. Let $U$ be a non-empty $k_1$-$\pi$ closed contained in $k_{2\alpha}(A) - A$. Now, $U \subseteq k_{2\alpha}(A)$ and $U \subseteq A$ and $A \subseteq U^c$. Since $U$ is $k_1$-$\pi$ closed, $U^c$ is $k_1$-$\pi$ open. Thus $k_{2\alpha}(A) \subseteq U^c$. Consequently, $U \subseteq [k_{2\alpha}(A)]^c$ and $U \subseteq k_{2\alpha}(A) \cap [k_{2\alpha}(A)]^c = \phi$. Therefore, $U = \phi$ and $k_{2\alpha}(A) - A$ contains no non-empty $k_1$-$\pi$ closed sets.

Proposition 3.5. Let $(X, k_1, k_2)$ be biČech closure space, For each $x$ in $X$, \{x\} is $k_1$-$\pi$ closed or \{x\}$^c$ is $(k_1, k_2)$-$\pi g\alpha$ closed set.

Proof. Let $(X, k_1, k_2)$ be biČech closure space. Suppose that \{x\} is not $k_1$-$\pi$ closed set, \{x\}$^c$ is not $k_1$-$\pi$ open set. Therefore the only $k_1$-$\pi$ open set containing \{x\}$^c$ is $X$. Thus \{x\}$^c \subseteq X$. Also $k_{2\alpha}[\{x\}^c] \subseteq k_{2\alpha}(X) = X$. Hence \{x\}$^c$ is a $\pi g\alpha$ closed set.

Proposition 3.6. Let $A$ be $(k_1, k_2)$-$\pi g\alpha$ closed set and if $A$ is $k_1$-$\pi$ open set then $A = k_{2\alpha}(A)$.

Proof. Let $A$ be $(k_1, k_2)$-$\pi g\alpha$ closed subset of a biČech closure space $(X, k_1, k_2)$ and let $A$ be $k_1$-$\pi$ open set. Then $k_{2\alpha}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $k_1$-$\pi$ open set in $X$. Since $A$ is $k_1$-$\pi$ open and $A \subseteq A$, we have $k_{2\alpha}(A) \subseteq A$. But always $A \subseteq k_{2\alpha}(A)$. Thus, $A = k_{2\alpha}(A)$.

Proposition 3.7. Let $A \subseteq Y \subseteq X$ and suppose that $A$ is $(k_1, k_2)$-$\pi g\alpha$ closed set in $(X, k_1, k_2)$. Then $A$ is $(k_1, k_2)$-$\pi g\alpha$ closed relative to $Y$.

Proof. Let $S$ be any $k_1$-$\pi$ open set in $Y$ such that $A \subseteq S$. Then $S = U \cap Y$ for some $U$ which is $k_1$-$\pi$ open set in $X$. Therefore $A \subseteq U \cap Y$ implies $A \subseteq U$.
Since $A$ is $(k_1,k_2)$-$\pi g\alpha$ closed set in $X$, we have $k_{2\alpha}(A) \subseteq U$. Hence $Y \cap k_{2\alpha}(A) \subseteq Y \cap U = S$. Thus $A$ is $\pi g\alpha$ -closed set relative to $Y$.

\section*{4 $(k_1,k_2)$-$\pi g\alpha$ open sets}

\textbf{Definition 4.1.} A subset $A$ in bi\v{C}ech closure space $(X, k_1, k_2)$ is called $(k_1,k_2)$-$\pi g\alpha$ open set if $A^c$ is $(k_1,k_2)$-$\pi g\alpha$ closed set in $(X, k_1, k_2)$.

\textbf{Proposition 4.2.} A subset $A$ of $(X, k_1, k_2)$ is called $(k_1,k_2)$-$\pi g\alpha$ open set if and only if $F \subseteq \text{int}_{k_2}(A)$ whenever $F$ is $k_1$-$\pi$ closed set and $F \subseteq A$.

\textit{Proof.} Suppose $A$ is $(k_1,k_2)$-$\pi g\alpha$ open set in $(X, k_1, k_2)$. Let $F$ be $k_1$-$\pi$ closed set and $F \subseteq A$. Then $F^c$ is $k_1$-$\pi$ open set and $A^c \subseteq F^c$. Since $A^c$ is $(k_1,k_2)$-$\pi g\alpha$ closed set, we have $k_{2\alpha}(A^c) \subseteq F^c$. This implies $F \subseteq [k_{2\alpha}(A^c)]^c = \text{int}_{k_2}(A)$. That is $F \subseteq \text{int}_{k_2}(A)$ whenever $F$ is $k_1$-$\pi$ closed set and $F \subseteq A$. Let $V$ be any $k_1$-$\pi$ open set in $X$ such that $A^c \subseteq V$. Thus $V^c \subseteq A$ and $V^c$ is $k_1$-semi closed. Therefore, $V^c \subseteq \text{int}_{k_2}(A)$. Hence we get $[\text{int}_{k_2}(A)]^c \subseteq V$. Implies $k_{2\alpha}(A^c) \subseteq V$ gives $A^c$ is $(k_1,k_2)$-$\pi g\alpha$ closed set. Thus $A$ is $(k_1,k_2)$-$\pi g\alpha$ open set.

\textbf{Corollary 4.3.} A subset $A$ of $(X, k_1, k_2)$ is $(k_1,k_2)$-$\pi g\alpha$ closed set, then $k_{2\alpha}(A) - A$ is $(k_1,k_2)$-$\pi g\alpha$ open set.

\textit{Proof.} Let $F$ be $k_1$-$\pi$ closed set such that $F \subseteq k_{2\alpha}(A) - A$. Then using proposition 3.6, $F = \emptyset$. Therefore $F \subseteq \text{int}_{2\alpha}\{k_{2\alpha}(A) - A\}$ and $k_{2\alpha}(A) - A$ is $(k_1,k_2)$-$\pi g\alpha$ open set.

\textbf{Proposition 4.4.} If $A$ and $B$ be $(k_1,k_2)$-$\pi g\alpha$ open set, then so is $A \cap B$. 

\hfill \Box
Proof. Let \( A^c \cup B^c \subseteq U \) where \( U \) is \( k_1-\pi \)-open. This implies \( A^c \subseteq U \) and \( B^c \subseteq U \), gives \( k_{2\alpha} (A^c) \subseteq U \) and \( k_{2\alpha} (B^c) \subseteq U \). Thus \( k_{2\alpha} (A^c) \cup k_{2\alpha} (B^c) \subseteq U \). Therefore \( k_{2\alpha} (A^c \cup B^c) \subseteq U \). Therefore \( A \cap B \) is \((k_1,k_2)\)-\( \pi g\alpha \) open set.

\[ \square \]

5 \((k_1,k_2)\)-\( \pi g\alpha - T_{\frac{1}{2}}\) biclosure space

Definition 5.1. A biclosure space \((X, k_1, k_2)\) is called a \( \pi g\alpha - T_{\frac{1}{2}}\) biclosure space if every \( \pi g\alpha \)-closed subset of \((X, k_1, k_2)\) is a \( k_1-\alpha \) closed.

Proposition 5.2. The biclosure space \((X, k_1, k_2)\) is a \( \pi g\alpha - T_{\frac{1}{2}}\) space iff every \( \{x\}\) of \( X \) is either \( k_1-\alpha \) open or \( k_2-\pi \) closed.

Proof. Let \( x \in X \) and suppose that \( \{x\}\) is not a \( k_2-\pi \)-closed subset of \( X \). Then \( X - \{x\}\) is not a \( k_2-\pi \) open subset of \( X \). The only \( \pi \)-open subsets of \((X, k_2)\) containing \( X - \{x\}\) is \( X \), hence \( X - \{x\}\) is a \((k_1,k_2)\)-\( \pi g\alpha \) closed subset of \( X \). Since \((X, k_1, k_2)\) is a \( \pi g\alpha - T_{\frac{1}{2}}\) biclosure space, \( X - \{x\}\) is \( k_1-\alpha \) closed subset of \( X \). Consequently \( \{x\}\) is \( k_1-\alpha \)-open subset of \( X \).

Conversely. Let \( A \) be \((k_1,k_2)\)-\( \pi g\alpha \) closed subset of \((X, k_1,k_2)\). Suppose \( x \notin A \), then \( \{x\} \subseteq X - A \) and we have \( A \subseteq X - \{x\}\). If \( \{x\}\) is \( k_1-\alpha \) open, then \( X - \{x\}\) is \( k_1-\alpha \) closed subset of \( X \) and we have \( k_{1\alpha}(A) \subseteq k_1(X - \{x\}) = X - \{x\} \) and thus \( x \notin k_{1\alpha}(A) \). If \( \{x\}\) is \( k_2-\pi \) closed subset of \( X \) then \( X - \{x\}\) is \( k_2-\pi \) open subset of \( X \). Since \( A \) is \((k_1,k_2)\)-\( \pi g\alpha \) closed, \( k_{2\alpha}(A) \subseteq X - \{x\}\). Therefore \( \{x\} \notin k_{1\alpha}(A) \) and we get \( k_{1\alpha}(A) \subseteq A \). Thus \( k_{1\alpha}(A) = A \) and \( A \) is \( k_1-\alpha \) closed in \( X \) and \( X \) is \( \pi g\alpha - T_{\frac{1}{2}} \) space.

\[ \square \]

6 Separation Axioms

In this section we introduce the concept of generalized \( \pi g\alpha \)-Hausdorff biclosure spaces, \( \pi g\alpha \)-regular biclosure spaces and study some of the seperation axioms. 

**Definition 6.1.** A biclosure space \((X, k_1, k_2)\) is said to be

1. \((k_1, k_2)\)-\(\pi g\alpha\)-Hausdorff space whenever \(x\) and \(y\) are distinct points of \(X\) there exist a \(k_1\)-\(\pi g\alpha\)-open subset \(U\) and \(k_2\)-\(\pi g\alpha\) open subset \(V\) of \(X\) such that \(x \in U\), \(y \in V\) and \(U \cap V = \phi\).

2. \((k_1, k_2)\)-\(\pi g\alpha\)-regular space if for any closed subset \(F\) of \((X)\) and any point \(x \in X - F\), there exist \(k_2\)-\(\pi g\alpha\) open subsets \(U\) and \(V\) of \(X\) such that \(x \in U\), \(F \subseteq V\) and \(U \cap V = \phi\).

**Definition 6.2.** Let \((X, u_1,u_2)\) and \((Y,v_1,v_2)\) be closure spaces. A map \(f : X \to Y\) is called \(\pi g\alpha\)-irresolute, if \(f^{-1}(F)\) is a \(\pi g\alpha\)-closed subset of \(X\) for every \(\pi g\alpha\)-closed subset \(F\) of \((Y,v)\).

Clearly, a map \(f : X \to Y\) is \(\pi g\alpha\)-irresolute if and only if \(f^{-1}(G)\) is a \(\pi g\alpha\)-open subset of \((X, u)\) for every \(\pi g\alpha\)-open subset \(G\) of \((Y,v)\).

**Definition 6.3.** Let \((X, u_1,u_2)\) and \((Y,v_1,v_2)\) be biclosure spaces and let \(i \in \{1, 2\}\). A map \(f : (X, u_i) \to (Y,v_i)\) is called \(i\)-\(\pi g\alpha\)-irresolute if the map \(f : (X, u_i) \to (Y,v_i)\) is \(\pi g\alpha\)-irresolute. A map \(f\) is called \(\pi g\alpha\)-irresolute if \(f\) is \(i\)-\(\pi g\alpha\)-irresolute for each \(i \in \{1, 2\}\).

**Proposition 6.4.** Let \((X, u_1,u_2)\) and \((Y, v_1,v_2)\) be biclosure spaces. Let \(f : (X, u_1,u_2) \to (Y, v_1,v_2)\) be injective and \(\pi g\alpha\)-irresolute. If \((Y, v_1,v_2)\) is a \((v_1,v_2)\)-\(\pi g\alpha\)-Hausdorff biclosure space, then \((X, u_1,u_2)\) is a \((u_1,u_2)\)-\(\pi g\alpha\)-Hausdorff space.

**Proof.** Let \(x\) and \(y\) be any two distinct points of \(X\). Then \(f(x)\) and \(f(y)\) are distinct points of \(Y\). Since \((Y, v_1,v_2)\) is a \((v_1,v_2)\)-\(\pi g\alpha\)-Hausdorff biclosure space, there exists a disjoint \(v_1\)-\(\pi g\alpha\)-open subset \(U\) of \((Y,v_1)\) and \(v_2\)-\(\pi g\alpha\)-open subset \(V\) of \((Y,v_2)\) containing \(f(x) \in U\) and \(f(y) \in V\) respectively. Since \(f\) is
$\pi g\alpha$-irresolute and $U \cap V = \phi$, $f^{-1}(U)$ is a $u_1-\pi g\alpha$ open subset of $X$ and $f^{-1}(V)$ is a $u_2-\pi g\alpha$ open subset of $X$ such that $f^{-1}(U) \cap f^{-1}(V) = \phi$ and $(X, u_1,u_2)$ is a $\pi g\alpha$-Hausdorff biclosure space.

Proposition 6.5. Let $(X, u_1,u_2)$ and $(Y, v_1,v_2)$ be biclosure spaces. Let $f : (X, u_1,u_2) \to (Y, v_1,v_2)$ be injective, closed and $\pi g\alpha$-irresolute. If $(Y, v_1,v_2)$ is a $\pi g\alpha$-regular biclosure space, then $(X, u_1,u_2)$ is a $\pi g\alpha$-regular biclosure space.

References


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