F-Semiprime Ideals in $\Gamma_N$—Semiring

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Abstract

We investigate here a general type of regularity and a general type of semiprime ideal i.e., F-semiprime ideal in a $\Gamma_N$—semiring $S$ with respect to different mappings viz., $+()$, $+^()$, $*()$, $*^()$, $S()$, $\Gamma()$.

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1 Introduction

The concept of $\Gamma$-semiring was introduced by M. M. K. Rao[2] in 1995 as a generalization of semirings. Dutta et al[1] introduced the notion of operator semirings of a $\Gamma$-semiring in 2002 and they obtained some nice correspondences between a $\Gamma$-semiring and its operator semirings. The $\Gamma$-semiring introduced
by Rao was one sided which was in the sense of Barnes. In 2008 Sardar et al[7]
introduced the notion of a both sided \( \Gamma \)-semiring in the sense of Nobusawa
which they called Nobusawa \( \Gamma \)-semiring or \( \Gamma_N \)-semiring. They also introduced
a general type of regularity (which they called F regularity)[4] and general type
of semiprime ideal (which they called F semiprime ideal) in semiring[4] in the
year 2009. Subsequently they extended these ideas to \( \Gamma \)-semiring[6, 5] in the
year 2010. They mainly studied the mappings \((x)^+, (x)^+/, (x)^*/(y)^*/(y)'\). Since more
mappings viz., \( +(), (+/(), (*)', (S(), \Gamma() \) exist for a Nobusawa \( \Gamma \)-semiring,
it is natural to investigate the behaviour of F-semiprime ideals under these
mappings. This paper is the outcome of this investigation.

For the convenience of the reader we recall the following diagram which
shows the domain and co-domain of the mappings mentioned above. Their
definitions are given in appropriate places.

\[
\begin{array}{c}
P(S) & \xrightarrow{S()} & P(S) \\
\Gamma() & \xrightarrow{P(\Gamma)} & P(R) \\
P(L) & \xrightarrow{P(\Gamma)} & P(L) \\
P(\Gamma) & \xrightarrow{\Gamma()} & P(R)
\end{array}
\]

P\( (S), P(\Gamma), P(L) \) and P\( (R) \) are respectively the power sets of S, \( \Gamma \), L and R.

2 Preliminaries

We recall the following so as to use in the sequel.

**Definition 2.1** [2] Let S and \( \Gamma \) be additive commutative monoids. S is
called a \( \Gamma \)-semiring if there exits a mapping \( S \times \Gamma \times S \rightarrow S \) (with \((m, \gamma, n) \rightarrow m \gamma n \) ) satisfying following condition for all \( m, n, p \in S \) and for all \( \gamma, \mu \in \Gamma \):

(i) \( m \gamma (n+p) = m \gamma n + m \gamma p \)
(ii) \( (m+n) \gamma p = m \gamma p + n \gamma p \)
(iii) \( m(\gamma + \mu)n = m \gamma n + m \mu n \)
(iv) \( m\gamma(n\mu p) = (m\gamma n)mp \)
(v) \( m\gamma 0 = 0\gamma m = 0 \), \( 0 \) is the zero element of \( S \)
(vi) \( m\theta n = 0 \), \( \theta \) is the zero element of \( \Gamma \).

If \( A \) and \( B \) are subsets of a \( \Gamma \)-semiring \( S \) and \( \Delta \subseteq \Gamma \), we denote by \( A\Delta B \), the subset of \( S \) consisting of all finite sums of the form \( \sum a_i\alpha_ib_i \) where \( a_i \in A, b_i \in B \) and \( \alpha_i \in \Delta \).

An additive subsemigroup \( I \) of a \( \Gamma \)-semiring \( S \) is called a left(right) ideal of \( S \) if \( S\Gamma I \subseteq I \) (\( I\Gamma S \subseteq I \)). If \( I \) is both a left ideal and right ideal then \( I \) is called a two-sided ideal or simply an ideal of \( S \).

**Definition 2.2** [1] Let \( S \) be a \( \Gamma \)-semiring and \( F \) be the free additive commutative semigroup generated by \( S \times \Gamma \). Then the relation \( \rho \) on \( F \), defined by
\[
\sum_{i=1}^{m}(x_i,\alpha_i)\rho \sum_{j=1}^{n}(y_j,\beta_j) \text{ if and only if } \sum_{i=1}^{m}x_i\alpha_i a = \sum_{j=1}^{n}y_j\beta_j a \text{ for all } a \in S \text{ (m, n} \in Z^+) \text{ is a congruence on } F. \text{ Congruence class containing } \sum_{i=1}^{m}(x_i,\alpha_i) \text{ is denoted by } \sum_{i=1}^{m}[x_i,\alpha_i]. \text{ Then } F/\rho \text{ is an additive commutative semigroup. Now } F/\rho \text{ forms a semiring with the multiplication defined by } \left( \sum_{i=1}^{m}[x_i,\alpha_i]\right) \left( \sum_{j=1}^{n}[y_j,\beta_j]\right) = \sum_{i,j}[x_i\alpha_i y_j,\beta_j]. \text{ This semiring is denoted by } L \text{ and called the left operator semiring of the } \Gamma \text{-semiring } S.

Dually the right operator semiring \( R \) of the \( \Gamma \)-semiring \( S \) has been defined where
\[
R = \left\{ \sum_{i=1}^{m}[\alpha_i,x_i]: \alpha_i \in \Gamma, x_i \in S, i = 1, 2, ..., m; m \in Z^+ \right\}
\]
and the multiplication on \( R \) is defined as \( \left( \sum_{i=1}^{m}[\alpha_i,x_i]\right) \left( \sum_{j=1}^{n}[\beta_j,y_j]\right) = \sum_{i,j}[\alpha_i,x_i\beta_j y_j]. \) Let \( S \) be a Nobusawa \( \Gamma \)-semiring and \( L \) and \( R \) respectively be the left and right operator semirings of the associated \( \Gamma \)-semiring \( S \). For any subset \( A \) of \( R \) we define \( ^* A = \left\{ \gamma \in \Gamma: [\gamma,S] = [\{\gamma\},S] \subseteq A \right\} \) and \( Q \subseteq L \), we define \( ^+ Q \) = \{ \gamma \in \Gamma: [S,\gamma] = [S,\{\gamma\}] \subseteq Q \}.

Again for \( \Theta \subseteq \Gamma \) we define \( ^* \Theta = \{ r \in R: \tau\Gamma \subseteq \Theta \} \) and \( ^{+} \Theta = \{ l \in L: \Gamma l \subseteq \Theta \} \).

Let \( A \subseteq S \) and \( \Phi \subseteq \Gamma \). Then \( \Gamma(A) = \{ \alpha \in \Gamma: SaS \subseteq A \} \), where \( SaS \) denotes the finite sums of the form \( \sum_{i=1}^{k}u_i\alpha v_i \), \( u_i, v_i \in S \) and \( S(\Phi) = \{ s \in S: \Gamma s \Gamma \subseteq \Phi \} \), where \( \Gamma s \Gamma \) denotes the finite sums of the form \( \sum_{j=1}^{p}\gamma_j s \lambda_j \), \( \gamma_j, \lambda_j \in \Gamma \).
Definition 2.3 Let $S$ be a $Γ-$semiring and $Γ$ be a $S$-semiring. If for $a, b, c \in S$ and $α, β, γ \in Γ$, $(αab)βc = a(αβ)c$ and $(αaβ)bγ = α(ab)γ = αa(βbγ)$ then $S$ is called a weak Nobusawa $Γ-$semiring. Also if the following condition hold: for all $s_1, s_2 \in S$, $s_1 α s_2 = s_1 β s_2$ implies $α = β$, then $S$ is called a Nobusawa $Γ-$semiring or simply $Γ_N$-semiring.

Definition 2.4 [1] Let $S$ be a $Γ-$semiring and $L$ and $R$ be its left operator semiring and right operator semiring respectively. If there exists an element $\sum_{i=1}^{m} [e_i, δ_i] \in L \sum_{j=1}^{n} [γ_j, x_j] \in R$ such that $\sum_{i=1}^{m} e_i δ_i a = a$ (respectively $\sum_{j=1}^{n} a γ_j x_j = a$) for all $a \in S$ then $S$ is said to have the left unity $\sum_{i=1}^{m} [e_i, δ_i]$ (respectively right unity $\sum_{j=1}^{n} [γ_j, x_j]$).

Definition 2.5 [4] [5] For every semiring $(Γ-$semiring) $S$, let us associate a mapping $F_S: S \to G(S, +)$, where $G(S, +)$ denotes the set of all submonoids of $(S, +)$. Then $\{F_S: S$ is a semiring $(Γ-$semiring)\} is called a regularity for semirings (respectively $Γ-$semirings) if the following conditions are satisfied
(a) if $Φ: S \to T$ is a semiring (respectively $Γ-$semiring) epimorphism then $F_T(Φ(a)) = Φ(F_S(a))$ for all $a \in S$,
(b) if $A$ is an ideal of $S$ and $a \in A$, then $F_A(a) \subseteq F_S(a)$,
(c) if $r, s \in S$ and $s \in F_S(r)$, then $F_S(r + s) \subseteq F_S(r)$.

Example 2.6 [4] [5] For a semiring $(Γ$-semiring) $S$ and $F_S: S \to G(S, +)$ be defined by
(i) $F_S(a) = aSa$ (respectively $F_S(a) = aΓSTa$),
(ii) $F_S(a) = SaS$ (respectively $F_S(a) = STaΓS$),
(iii) $F_S(a) = aS$ (respectively $F_S(a) = aΓS$),
(iv) $F_S(a) = Sa$ (respectively $F_S(a) = STa$),
(v) $F_S(a) = SaSaS$ (respectively $F_S(a) = STaΓSTaΓS$), $F_S$ gives rise to a regularity for semirings (respectively $Γ-$semirings). The regularities (i), (ii), (iii), (iv) and (v) are respectively called Von Neumann regularity or simply regularity, $λ$-regularity, right $D$-regularity, left $D$-regularity and $f$-regularity in semirings ($Γ-$semirings).

From the Definition 2.3 it follows that for $Γ_N$-semiring $S$, $S$ is $Γ-$semiring and $Γ$ is a $S$-semiring.

3 Main Results

If otherwise not mentioned in this section $S$ will denote a $Γ_N-$semiring and $L, R$ respectively as the left, right operator semirings of the corresponding $Γ-$semiring $S$. 
Definition 3.1 A regularity \( \{ F_X : X \text{ is a semiring or } \Gamma_N-\text{semiring} \} \) is said to satisfy condition D if the following conditions are satisfied:

(a) \( F_\Gamma(\Gamma s \Gamma) \subseteq \Gamma F_\Gamma(s \Gamma) \) for \( s \in S \); ii) \( F_\Gamma(S \alpha S) \subseteq SF_\Gamma(\alpha)S \) for \( \alpha \in \Gamma \).
(b) \( F_\Gamma(r \Gamma) \subseteq F_R(r \Gamma) \) for \( r \in R \); ii) \( F_\Gamma(\Gamma l) \subseteq \Gamma F_L(l) \) for \( l \in L \).
(c) \( F_R([\alpha, S]) = [F_\Gamma(\alpha), S] \) for \( r \in R \); ii) \( F_L([S, \alpha]) = [S, F_\Gamma(\alpha)] \) for \( l \in L \).

Proposition 3.2 (A) Von numann regularity satisfies the condition D, (B) Left D-regularity satisfies the condition D, (C) Right D-regularity satisfies the condition D, (D) \( \lambda \)-regularity satisfies the condition D, (E) \( f \)-regularity satisfies the condition D.

Proof. (A) Let \( F \) denote the Von numann regularity.

a) i) Let \( s \in S \). Then \( F_\Gamma(\Gamma s \Gamma) = (\Gamma s \Gamma)ST S(\Gamma s \Gamma) \)
\( \subseteq \Gamma s[\Gamma, S][\Gamma, S] \Gamma s \Gamma \)
\( \subseteq \Gamma sR^2 s \Gamma \)
\( \subseteq \Gamma sR \Gamma s \Gamma \)
\( \subseteq \Gamma s[\Gamma, S] \Gamma s \Gamma \)
\( \Gamma (s \Gamma ST s) \Gamma \)
\( = \Gamma F_\Gamma(s \Gamma) \).

ii) Let \( \alpha \in \Gamma \). Then \( F_S(S \alpha S) = (S \alpha S) \Gamma ST S(S \alpha S) \)
\( \subseteq S \alpha[\Gamma, S][\Gamma, S] S \alpha S \)
\( \subseteq S \alpha L^2 S \alpha S \)
\( \subseteq S \alpha L \alpha S \alpha S \)
\( \subseteq S \alpha[\Gamma, S] S \alpha S \)
\( \subseteq S(\alpha \Gamma ST S \alpha) S \)
\( = SF_\Gamma(\alpha) S. \)

b)i) Let \( r \in R \). Then \( F_\Gamma(r \Gamma) = (r \Gamma)ST S(r \Gamma) \)
\( \subseteq r[\Gamma, S][\Gamma, S] r \Gamma \)
\( \subseteq r R^2 r \Gamma \)
\( \subseteq r R r \Gamma \)
\( = F_R(r \Gamma) \).

ii) Let \( l \in L \). Then \( F_\Gamma(\Gamma l) = (\Gamma l)ST S(\Gamma l) \)
\( \subseteq \Gamma l[S, \Gamma][S, \Gamma] l \)
\( \subseteq \Gamma lL^2 l \)
\( \subseteq \Gamma l \Gamma l \)
\( = \Gamma F_L(l) \).

(c i) Let \( \alpha \in \Gamma \). Then \( F_R([\alpha, S]) = [\alpha, S] R[\alpha, S] \)
\( = [\alpha, S][\Gamma, S][\alpha, S] \)
\( = [\alpha \Gamma ST S \alpha, S] \)
\( = [F_\Gamma(\alpha), S]. \)

ii) Let \( \alpha \in \Gamma \). Then \( F_L([S, \alpha]) = [S, \alpha] L[S, \alpha] \)
\( = [S, \alpha][\Gamma, S][S, \alpha] \)
\( = [S, \alpha \Gamma ST S \alpha]. \)
By applying similar argument we obtain the other cases. ■

The following examples shows that not every regularity satisfies condition D.

**Example 3.3** Let $S=\Gamma=Z_0^+$. Then $S$ is a $\Gamma_N$-semiring with respect to usual addition and multiplication. Suppose $F_S(a) = \{x + a\alpha x : x \in S\}$ for some fixed $\alpha \in \Gamma$, where $a \in S$ ( hence $F_{\Gamma}(\alpha) = \{\gamma + \alpha s\gamma : \gamma \in \Gamma\}$ for some fixed $s \in S$, where $\alpha \in \Gamma$). Then $F$ gives rise to a regularity. Now if we take $r = [1,1] \in R$ and $\alpha = 2 \in \Gamma$, then $F_{\Gamma}(r\Gamma) = Z_0^+$. Again $F_{R}(r\Gamma) = F_{R}([1,1])Z_0^+ = 2Z_0^+$. Hence $F_S(r\Gamma) \not\subseteq F_{R}(r\Gamma)$.

**Definition 3.4** [4][6] Let $\{F_S: S$ is a semiring$; \Gamma_N$-semiring$\}$ be a regularity for $\Gamma_N$-semirings. An ideal $I$ of a semiring$; \Gamma_N$-semiring$\}$ $S$ is called $F$-semiprime if $F_S(r) \subseteq I$ implies $r \in I$ where $r \in S$.

Henceforth $S$ will denote a $\Gamma_N$-semiring with unities and $L,R$ be the left and right operator semirings of $S$ respectively and $\{F_X: X$ is a semiring or $\Gamma_N$-semiring$\}$ will be a regularity for semirings or $\Gamma_N$-semirings satisfying condition D.

**Proposition 3.5** Suppose $P$ is an $F$-semiprime ideal of $S$. Then $\Gamma(P)$ is an $F$-semiprime ideal of $\Gamma$.

**Proof.** Let $P$ be an $F$-semiprime ideal of $S$. Then $P$ is an ideal of $S$. Hence $\Gamma(P)$ is an ideal of $\Gamma$ (cf. Theorem 4.1[7]). Let $\alpha \in \Gamma$ and $F_{\Gamma}(\alpha) \subseteq \Gamma(P)$. Then $SF_{\Gamma}(\alpha)S \subseteq \Gamma(P)S$.

$\Rightarrow SF_{\Gamma}(\alpha)S \subseteq P$ [follows from the definition of $\Gamma(P)$],

$\Rightarrow F_S(S\alpha S) \subseteq P$ [since $F$ satisfies condition D],

$\Rightarrow S\alpha S \subseteq P$ [since $P$ is an $F$-semiprime ideal of $S$],

$\Rightarrow \alpha \in \Gamma(P)$ [follows from the definition of $\Gamma(P)$].

Consequently, $\Gamma(P)$ is an $F$-semiprime ideal of $\Gamma$. ■

**Proposition 3.6** Suppose $\Phi$ is an $F$-semiprime ideal of $\Gamma$. Then $S(\Phi)$ is an $F$-semiprime ideal of $S$.

**Proof.** Let $\Phi$ be an $F$-semiprime ideal of $\Gamma$. Then $\Phi$ is an ideal of $\Gamma$. Hence $S(\Phi)$ is an ideal of $S$ (cf. Note bellow Theorem 4.1[7]). Let $a \in S$ and $F_S(a) \subseteq S(\Phi)$. Then $SF_{\Gamma}(\alpha)S \subseteq \Gamma(P)S$.

$\Rightarrow \Gamma F_S(a) \Gamma \subseteq \Phi$ [follows from the definition of $S(\Phi)$],

$\Rightarrow F_{\Gamma}(\Gamma a \Gamma) \subseteq \Phi$ [since $F$ satisfies condition D],

$\Rightarrow \Gamma a \Gamma \subseteq \Phi$ [since $\Phi$ is an $F$-semiprime ideal of $\Gamma$],

$\Rightarrow a \in S(\Phi)$ [follows from the definition of $S(\Phi)$].

Consequently, $S(\Phi)$ is an $F$-semiprime ideal of $S$. ■
Proposition 3.7 Let $P$ be an $F$-semiprime ideal of $R$. Then $^*P$ is an $F$-semiprime ideal of $\Gamma$.

Proof. Let $P$ be an $F$-semiprime ideal of $R$. Then $P$ is an ideal of $R$. Hence $^*P$ is an ideal of $\Gamma$ (cf. Note below Theorem 4.11[7]). Let $\alpha \in \Gamma$ and $F_{\Gamma}(\alpha) \subseteq ^*P$. Then $[F_{\Gamma}(\alpha), S] \subseteq [^*P, S]$.

$\Rightarrow [F_{\Gamma}(\alpha), S] \subseteq P$ [follows from the definition of $^*P$],

$\Rightarrow F_R([\alpha, S]) \subseteq P$ [since $F$ satisfies condition D],

$\Rightarrow [\alpha, S] \subseteq P$ [since $P$ is an $F$-semiprime ideal of $R$],

$\Rightarrow \alpha \in ^*P$ [follows from the definition of $^*P$].

Consequently, $^*P$ is an $F$-semiprime ideal of $\Gamma$.

Analogously we can prove the following result for left operator semiring.

Proposition 3.8 Let $Q$ be an $F$-semiprime ideal of $L$. Then $^+Q$ is an $F$-semiprime ideal of $\Gamma$.

Proposition 3.9 Let $\Phi$ be an $F$-semiprime ideal of $\Gamma$. Then $^*\Phi$ is an $F$-semiprime ideal of $R$.

Proof. Let $\Phi$ be an $F$-semiprime ideal of $\Gamma$. Then $\Phi$ is an ideal of $R$. Hence $^*\Phi$ is an ideal of $\Gamma$ (cf. Note below Theorem 4.11[7]). Let $r \in R$ and $F_{\Gamma}(r) \subseteq ^*\Phi$. Then $F_R(r) \subseteq ^*\Phi$.

$\Rightarrow F_R(r) \Gamma \subseteq \Phi$ [follows from the definition of $^*\Phi$],

$\Rightarrow F_{\Gamma}(r \Gamma) \subseteq \Phi$ [since $F$ satisfies condition D],

$\Rightarrow r \Gamma \subseteq \Phi$ [since $\Phi$ is an $F$-semiprime ideal of $\Gamma$],

$\Rightarrow r \in ^*\Phi$ [follows from the definition of $^*\Phi$].

Consequently, $^*\Phi$ is an $F$-semiprime ideal of $R$.

Analogously we can prove the following result for left operator semiring.

Proposition 3.10 Let $\Phi$ be an $F$-semiprime ideal of $\Gamma$. Then $^+\Phi$ is an $F$-semiprime ideal of $L$.

To conclude the paper we obtain below various inclusion preserving bijections for $F$-semiprime ideals.

Theorem 3.11 Let $S$ be a $\Gamma_N$-semiring with unities and $L$, $R$ be respectively the left and right operator semirings of the corresponding $\Gamma$-semiring $S$. Suppose $\{F_X : X$ is a semiring or $\Gamma$-semiring $\}$ is a regularity for semirings or $\Gamma$-semirings satisfying the condition D. Then there exists

(i) an inclusion preserving bijection between the set of $F$-semiprime ideals of $S$ and that of $\Gamma$ via the mapping $I \mapsto \Gamma(I)$,

(ii) an inclusion preserving bijection between the set of $F$-semiprime ideals of $\Gamma$ and that of $R$ via the mapping $P \mapsto ^*P$,

(iii) an inclusion preserving bijection between the set of $F$-semiprime ideals of $\Gamma$ and that of $L$ via the mapping $Q \mapsto ^+Q$. 

**Proof.** (i) We know that \( I \mapsto \Gamma(I) \) is an inclusion preserving bijection between the set of ideals of \( S \) and that of \( \Gamma \). Now since an \( F \)-semiprime ideal is firstly an ideal, the rest of the theorem follows from Propositions 3.5 and 3.6.

(ii) By Theorem 4.15 of [7], \( P \mapsto {}^*P \) is an inclusion preserving bijection between the set of ideals of \( \Gamma \) and that of \( R \). The rest of the theorem follows from Propositions 3.7 and 3.8.

(iii) By Theorem 4.15 of [7], \( Q \mapsto {}^+Q \) is an inclusion preserving bijection between the set of ideals of \( \Gamma \) and that of \( L \). Then the theorem follows from Proposition 3.9 and 3.10.

**Concluding Remark.** In this paper we discuss a general type of regularity of a \( \Gamma_N \)-semiring \( S \) and corresponding semiprime ideals viz., \( F \)-semiprime ideals. Since corresponding to every \( \Gamma_N \)-semiring there is a matrix semiring \( S_2 = \begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix} \) defined by Sardar et al[3], it is very natural to investigate, as a sequel to this paper, the interrelation between \( S \) and \( S_2 \) in terms of those regularities and corresponding \( F \)-semiprime ideals.

**References**


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