Schur Harmonic Convexity of Gini Means

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Abstract. In this paper, necessary and sufficient conditions for the Schur harmonic convex and Schur harmonic concave of Gini means are given.

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1. Introduction

Let $p, q \in \mathbb{R}$ and $a, b > 0$. The Gini means are defined as [13].

\begin{equation}
G_{p,q}(a, b) = \begin{cases}
\left( \frac{a^p + b^p}{a^q + b^q} \right)^{1/(p-q)}, & p \neq q, \\
\exp\left( \frac{ap \ln a + bq \ln b}{a^p + b^p} \right), & p = q.
\end{cases}
\end{equation}

It is easy to see that the Gini means $G_{p,q}(a, b)$ are continuous on the domain $\{(a, b; p, q) : a, b \in \mathbb{R}_+; p, q \in \mathbb{R}\}$ and differentiable with respect to $(a, b) \in (0, \infty) \times (0, \infty)$ for fixed $p, q \in \mathbb{R}$. Also, Gini means are symmetric with respect to $a, b$ and $p, q$.

There has been a lot of literature such as [8, 9, 10, 11, 20, 21, 25, 26, 27, 30, 36, 43, 44, 45], and the related references therein about inequalities and properties of Gini means.

In recent years, the Schur convexity, Schur geometrically convexity and Schur harmonic convexity of $G_{p,q}(a, b)$ have attracted the attention of a considerable number of mathematicians [3, 4, 5, 14, 19, 28, 29, 31, 33, 34, 38, 39, 40]. Sándor [31] proved that the Gini means $G_{p,q}(a, b)$ are Schur convex on $(-\infty, 0] \times (-\infty, 0]$ and Schur concave on $[0, \infty) \times [0, \infty)$ with respect to
(p, q) for fixed a, b > 0 with a ≠ b. Yang [46] improved Sándor’s result and proved that Gini means \( G_{p,q}(a, b) \) are Schur convex with respect to (p, q) for fixed a, b > 0 if and only if \( p + q < 0 \) and Schur concave if and only if \( p + q > 0 \). Wang and Zhang [38, 39] shown that Gini means \( G_{p,q}(a, b) \) are Schur convex with respect to \((a, b) \in (0, \infty) \times (0, \infty)\) if and only if \( p + q \geq 1 \), \( p, q \geq 0 \) and Schur concave if and only if \( p + q \leq 1 \), \( p \leq 0 \) or \( p + q \leq 1 \), \( q \leq 0 \).

Gu and Shi [14, 34] also discussed the Schur convexity. Recently, Chu and Xia [5] also proved the same result as Wang and Zhang’s.

For the Schur geometrically convexity, Wang and Zhang [38, 39] proved that Gini means \( G_{p,q}(a, b) \) are Schur geometrically convex with respect to \((a, b) \in (0, \infty) \times (0, \infty)\) if and only if \( p + q \geq 0 \) and Schur geometrically concave if \( p + q \leq 0 \).

Gu and Shi [14, 34] also investigated the Schur geometrically convexities of Lehmer means \( G_{p,1-p}(a, b) \) and Gini mans \( G_{p,q}(a, b) \), respectively.

For the Schur harmonic convexity, Xia and Chu [40] gave an necessary and sufficient condition for Lehmer means \( G_{p,1-p}(a, b) \).

**Theorem 1** ([40, Theorem 1.1]). The Lehmer mean values \( G_{p,p-1}(a, b) \) are Schur harmonic convex with respect to \((a, b) \in (0, \infty) \times (0, \infty)\) if and only if \( p \geq 0 \) and Schur harmonic concave with respect to \((a, b) \in (0, \infty) \times (0, \infty)\) if and only if \( p \leq 0 \).

The purpose of this paper is to investigate the Schur harmonic convexity of Gini means \( G_{p,q}(a, b) \), of which the idea is to find the necessary conditions from Lemma 2, and then prove these conditions are sufficient. Our main result is as follows.

**Theorem 2.** (1) Gini means \( G_{p,q}(a, b) \) are Schur harmonic convex with respect to \((a, b) \in (0, \infty) \times (0, \infty)\) if and only if \( p + q + 1 \geq 0 \) and \( \max(p, q) \geq 0 \).

(2) Gini means \( G_{p,q}(a, b) \) are Schur harmonic concave with respect to \((a, b) \in (0, \infty) \times (0, \infty)\) if and only if \( p + q + 1 \leq 0 \) and \( \max(p, q) \leq 0 \).

2. Definitions and Lemmas

Schur convexity was introduced by Schur in 1923 [22], and it has many important applications in analytic inequalities [2, 15, 47], linear regression [35], graphs and matrices [7], combinatorial optimization [17], information-theoretic topics [12], Gamma functions [23], stochastic orderings [32], reliability [16], and other related fields. Recently, Anderson et al. [1] discussed an attractive class of inequalities, which arise from the notation of harmonic convexity.

For convenience of readers, we recall some definitions as follows.

**Definition 1.** [22, 37] Let \( x = (x_1, x_2, ..., x_n) \) and \( y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n(n \geq 2) \).
(i) $x$ is said to be majorized by $y$ (in symbol $x \prec y$) if

\begin{equation}
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \text{ for } 1 \leq k \leq n-1, \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,
\end{equation}

where $x_1 \geq x_2 \cdots \geq x_n$ and $y_1 \geq y_2 \cdots \geq y_n$ are rearrangements of $x$ and $y$ in a decreasing order.

(ii) $x \geq y$ means $x_i \geq y_i$ for all $i = 1, 2, \cdots, n$. Let $\Omega \subset \mathbb{R}^n (n \geq 2)$. The function $\phi: \Omega \to \mathbb{R}$ is said to be increasing if $x \geq y$ implies $\phi(x) \geq \phi(y)$. $\phi$ is said to be decreasing if and only if $-\phi$ is increasing.

(iii) $\Omega \subset \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \cdots, \alpha x_n + \beta y_n) \in \Omega$ for all $x$ and $y$, where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

(iv) Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a set with nonempty interior. Then $\phi: \Omega \to \mathbb{R}$ is said to be Schur convex if $x \prec y$ on $\Omega$ implies $\phi(x) \leq \phi(y)$. $\phi$ is said to be Schur concave if $-\phi$ is Schur convex.

**Definition 2.** [24, 48] Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n (n \geq 2)$. Denote

\begin{equation}
\frac{1}{x} = \left( \frac{1}{x_1} \frac{1}{x_2} \ldots \frac{1}{x_n} \right) \quad \text{and} \quad \frac{1}{y} = \left( \frac{1}{y_1} \frac{1}{y_2} \ldots \frac{1}{y_n} \right).
\end{equation}

(i) $\Omega \subset \mathbb{R}^n$ is called a harmonic convex set if $(\frac{2x_1y_1}{x_1+y_1}, \cdots, \frac{2x_ny_n}{x_n+y_n}) \in \Omega$ for all $x$ and $y$.

(ii) Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a set with nonempty interior. Then function $\phi: \Omega \to \mathbb{R}$ is said to be Schur harmonic convex on $\Omega$ if $\frac{1}{x} \prec \frac{1}{y}$ on $\Omega$ implies $\phi(x) \leq \phi(y)$. $\phi$ is said to be Schur harmonic concave if $-\phi$ is Schur harmonic convex.

**Definition 3.** [22](i) $\Omega \subset \mathbb{R}^n (n \geq 2)$ is called symmetric set, if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix $P$.

(ii) The function $\phi: \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix $P$, $\phi(Px) = \phi(x)$ for all $x \in \Omega$.

The following well-known result was proved by Marshall and Olkin [22] (also see [37]).

**Lemma 1.** Let $\Omega \subset \mathbb{R}^n$ be a symmetric set with nonempty interior $\Omega^0$ and $\phi: \Omega \to \mathbb{R}$ be continuous on $\Omega$ and differentiable in $\Omega^0$. Then $\phi$ is Schur convex (Schur concave) on $\Omega$ if and only if $\phi$ is symmetric on $\Omega$ and

\begin{equation}
(x_1 - x_2) \left( \frac{\partial \phi}{\partial x_1} - \frac{\partial \phi}{\partial x_2} \right) \geq (\leq) 0
\end{equation}

holds for any $x = (x_1, x_2, \ldots, x_n) \in \Omega^0$.

For the Schur harmonic convexity, there has similar result [6, 40, 41].
Lemma 2. Let \( \Omega \subset \mathbb{R}^n \) be a symmetric set with a nonempty interior harmonic convex set \( \Omega^0 \). Let \( \phi : \Omega \to \mathbb{R} \) be continuous on \( \Omega \) and differentiable in \( \Omega^0 \). Then \( \phi \) is Schur harmonic convex (Schur harmonic concave) on \( \Omega \) if and only if \( \phi \) is symmetric on \( \Omega \) and

\[
(x_1 - x_2) \left( x_1^2 \frac{\partial \phi}{\partial x_1} - x_2^2 \frac{\partial \phi}{\partial x_2} \right) \geq (\leq) 0
\]

holds for any \( x = (x_1, x_2, \ldots, x_n) \in \Omega^0 \).

3. Proof of Main Result

Lemma 3. Let \( G = G_{p,q} := G_{p,q}(a,b) \) defined by (1.1) and

\[
A = p + q + 1, \quad B = p - q + 1, \quad C = p - q - 1.
\]

Then

\[
\Delta := (a - b) \left( a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} \right) = \frac{G \sqrt{ab} (a - b)}{2 \cosh pt \cosh qt} g_{p,q}(t),
\]

where

\[
g_{p,q}(t) = \begin{cases} 
(p-q) \sinh At + \sinh Bt + q \sinh Ct & \text{if } p \neq q, \\
\sinh(2p + 1)t + \sinh t + 2pt \cosh t & \text{if } p = q,
\end{cases}
\]

\( t = \ln \sqrt{a/b} \) and \( g(p,q,t) := g_{p,q}(t) \in C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+) \).

Proof. For \( p \neq q \), some simple partial derivative calculations yield

\[
\frac{\partial \ln G}{\partial a} = \frac{1}{G} \frac{\partial G}{\partial a} = \frac{1}{p - q} \left( \frac{pa^{p-1}}{a^p + b^p} - \frac{qa^{q-1}}{a^q + b^q} \right),
\]

\[
\frac{\partial \ln G}{\partial b} = \frac{1}{G} \frac{\partial G}{\partial b} = \frac{1}{p - q} \left( \frac{pb^{p-1}}{a^p + b^p} - \frac{qb^{q-1}}{a^q + b^q} \right),
\]

hence,

\[
a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} = \frac{G}{p - q} \left( p \frac{a^{p+1} - b^{p+1}}{a^p + b^p} - q \frac{a^{q+1} - b^{q+1}}{a^q + b^q} \right).
\]

Substituting \( \ln \sqrt{a/b} = t \) and using \( \sinh x = \frac{1}{2}(e^x - e^{-x}) \), \( \cosh x = \frac{1}{2}(e^x + e^{-x}) \), the right hand side above can be written as

\[
a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} = \frac{G \sqrt{ab}}{p - q} \left( \frac{p \sinh(p + 1)t}{\cosh pt} - \frac{q \sinh(q + 1)t}{\cosh qt} \right)
\]

\[
= \frac{G \sqrt{ab}}{2 \cosh pt \cosh qt} \frac{2p \sinh(p + 1)t \cosh qt - 2q \sinh(q + 1)t \cosh pt}{p - q}.
\]
Using the “product into sum” formula for hyperbolic functions and (3.1), we have
\[
(3.6) \frac{\partial^2 G}{\partial a \partial b} - b^2 \frac{\partial^2 G}{\partial b^2} = \frac{G \sqrt{ab}}{2 \cosh pt \cosh qt} \frac{(p - q) \sinh At + p \sinh Bt + q \sinh Ct}{p - q} \]
\[
= \frac{G \sqrt{ab}}{2 \cosh pt \cosh qt} \varphi_{p,q}(t).
\]
For \(p = q\), note \(G_{p,q}(a, b)\) can be expressed as
\[
(3.5) \quad G_{p,q}(a, b) = \int_0^1 Z(tp + (1 - t)q; a, b)dt, \quad \text{where} \quad Z(t; a, b) = e^{a't \ln a + b't \ln b},
\]
the integrand in (3.5) has continuous partial derivatives of any order with respect to \(p, q, a, b\) on \(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+\), hence \(G_{p,q}(a, b) \in C^\infty\). Thus
\[
\frac{\partial G_{p,p}}{\partial a} = \lim_{q \to p} \frac{\partial G_{p,q}}{\partial a}, \quad \frac{\partial G_{p,p}}{\partial b} = \lim_{q \to p} \frac{\partial G_{p,q}}{\partial b}.
\]
From (3.4) and by a limit calculation we obtain
\[
\left( a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} \right) \bigg|_{q=p} = a^2 \frac{\partial \left( \lim_{q \to p} G \right)}{\partial a} - b^2 \frac{\partial \left( \lim_{q \to p} G \right)}{\partial b} \]
\[
= \lim_{q \to p} \left( a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} \right) \]
\[
= \lim_{q \to p} \left( \frac{G \sqrt{ab}}{2 \cosh pt \cosh qt} \right) \frac{(p - q) \sinh At + p \sinh Bt + q \sinh Ct}{p - q} \]
\[
= \frac{G \sqrt{ab}}{2 \cosh pt \cosh qt} \varphi_{p,p}(t).
\]
Lastly, let us prove \(g(p, q, t) := g_{p,q}(t) \in C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+)\).
It is easy to see
\[
g(p, q, t) = \begin{cases} \frac{1}{p-q} \int_q^p (\sinh A_1 t + (u - q)t \cos A_1 t + \sinh B_1 t + ut \sinh B_1 t + qt \cosh C_1 t)du, & \text{if } p \neq q, \\ \sinh(2p + 1)t + \sinh t + 2pt \cosh t & \text{if } p = q, \end{cases}
\]
where \(A_1 = u + q + 1, \ B_1 = u - q + 1, \ C_1 = u - q - 1.\) With \(u = pv + (1 - v)q,\)
then
\[
(3.6) \quad g(p, q, t) = \int_0^1 \left( \sinh A_2 t + v(p - q)t \cos A_2 t + \sinh B_2 t + (pv + (1 - v)q)t \sinh B_2 t + qt \cosh C_2 t \right)dv,
\]
where \(A_2 = pv + (2 - v)q + 1, \ B_2 = (p - q)v + 1, \ C_2 = (p - q)v - 1.\)
It is clear that the integrand in (3.6) has continuous partial derivatives of any order with respect to $p, q, t$ on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$, which follows $g(p, q, t) \in C^{\infty}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+)$. This lemma is proved.

**Lemma 4.** Let $g(t) := g_{p,q}(t)$ and $\beta = \max(|A|, |B|, |C|)$. Then

\[
\lim_{t \to 0} \frac{g(t)}{t} = \lim_{t \to 0} g'(t) = 2(p + q + 1), \tag{3.7}
\]

\[
\lim_{t \to \infty} \frac{2\beta g(t)}{e^{\beta t}} = \lim_{t \to \infty} \frac{2g'(t)}{e^{\beta t}} = \begin{cases} p + q + 1 & \text{if } p > q > 0 \text{ or } -1 > p > q, \\
-q^2/(p-q) & \text{if } p = -1 > q, \\
2(p+1) & \text{if } p > q = 0, \\
p(p-q+1)/(p-q) & \text{if } p > -1, q < 0, p > q, \\
2p+1 & \text{if } p > 0 \text{ or } p < -1, \\
2 & \text{if } p = 0, \\
-\infty & \text{if } -1 \leq p < 0, \end{cases} \tag{3.8}
\]

\[
\lim_{t \to \infty} \frac{2\beta g(t)}{e^{\beta t}} = \lim_{t \to \infty} \frac{2g'(t)}{e^{\beta t}} = \begin{cases} 2p+1 & \text{if } p > 0 \text{ or } p < -1, \\
2 & \text{if } p = 0, \\
-\infty & \text{if } -1 \leq p < 0, \end{cases} \tag{3.9}
\]

**Proof.** A simple calculation yields

\[
g'(t) = \begin{cases} \frac{(p-q)A \cosh At + pB \cosh Bt + qC \cosh Ct}{p-q} & \text{if } p \neq q, \\
(2p+1) \cosh(2p+1)t + (2p+1) \cosh t + 2pt \sinh t & \text{if } p = q. \end{cases} \tag{3.10}
\]

(3.7) easily follows from a simple limit calculation. For $p \neq q$, since $g(0) = 0$, applying L’Hospital’s rule yields

\[
\lim_{t \to 0} \frac{g(t)}{t} = \lim_{t \to 0} g'(t) = \lim_{t \to 0} \frac{(p-q) \sinh At + p \sinh Bt + q \sinh Ct}{p-q} = \frac{(p-q)A + pB + qC}{p-q} = 2(p + q + 1).
\]

Likewise, for $p = q$, we have

\[
\lim_{t \to 0} \frac{g(t)}{t} = \lim_{t \to 0} g'_{p,p}(t) = \lim_{q \to p} \left( \lim_{t \to 0} \frac{g_{p,q}(t)}{t} \right) = \lim_{q \to p} (2(p + q + 1)) = 2(2p + 1).
\]
(3.9) easily follows from the limit relation

\[
\lim_{t \to \infty} \frac{2 \cosh \alpha t}{e^{\beta t}} = \begin{cases} 
1 & \text{if } \beta = |\alpha|, \\
0 & \text{if } \beta > |\alpha|.
\end{cases}
\]

In the case of \( p > q \), we have

\[
(p - q) \lim_{t \to \infty} \frac{2g(t)}{e^{\beta t}} = (p - q) \lim_{t \to \infty} \frac{2g'(t)}{e^{\beta t}}
= \lim_{t \to \infty} \frac{(p - q)A \cosh At + pB \cosh Bt + qC \cosh Ct}{e^{\beta t}}
= \begin{cases} 
(1) (p - q)A & \text{if } |p + q + 1| > |p - q + 1|, |p - q - 1|, \\
(2) (p - q)A + pB & \text{if } |p + q + 1| = |p - q + 1| > |p - q - 1|, \\
(3) (p - q)A + qC & \text{if } |p + q + 1| = |p - q - 1| > |p - q + 1|, \\
(4) pB & \text{if } |p - q + 1| > |p + q + 1|, |p - q - 1|, \\
(5) pB + qC & \text{if } |p - q + 1| = |p - q - 1| > |p + q + 1|, \\
(6) (p - q)A + pB & \text{if } |p + q + 1| > |p - q - 1|, \\
qC & \text{if } |p - q - 1| > |p + q + 1|, |p - q + 1|, \\
(7) qC & \text{if } |p - q - 1| = |p + q + 1| > |p - q + 1|, \\
(8) (p - q)A + qC & \text{if } |p + q - 1| = |p + q + 1| > |p - q + 1|, \\
(9) pB + qC & \text{if } |p - q - 1| = |p - q + 1| > |p + q + 1|, \\
(10) (p - q)A + pB + qC & \text{if } |p + q + 1| = |p - q - 1| = |p - q + 1|.
\end{cases}
\]

In all cases above, the case (2)=(6), case (3)=(8), case (5)=(9); while the case (3), (7) and case (5), (10) are required to satisfy \( p < q \) and \( p = q \), respectively. After removing three the same cases, that are case (6), (8), (9); and two cases of being required \( p < q \), that are case (3), (7); and two cases of being required \( p = q \), that are case (5), (10), and by rearrangements, we obtain

\[
(p - q) \lim_{t \to \infty} \frac{2g(t)}{e^{\beta t}}
= \begin{cases} 
(1) (p - q)(p + q + 1) & \text{if } (p + 1)q > 0, \|p(q + 1)\| > 0, p > q, \\
(2) 2p(p + 1) - q(p + q + 1) & \text{if } (p + 1)q = 0, p > q, \\
(4) p(p - q + 1) & \text{if } (p + 1)q < 0, p > q, \\
(p - q)(p + q + 1) & \text{if } p > q > 0 \text{ or } -1 > p > q, \\
-pq^2 & \text{if } p = -1 > q, \\
2p(p + 1) & \text{if } p > q = 0, \\
p(p - q + 1) & \text{if } p > -1, q < 0, p > q.
\end{cases}
\]
Divided by \((p - q)\) in the above limit relation, the first part of (3.9) follows. In the case of \(p = q\). Since \(g(p, q, t) \in C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+)\)

\[
\lim_{t \to \infty} \frac{2\beta g(t)}{e^{\beta t}} = \lim_{t \to \infty} \frac{2\beta g_{p,p}(t)}{e^{\beta t}} = \lim_{q \to p,q < p} \left( \lim_{t \to \infty} \frac{2\beta g_{p,q}(t)}{e^{\beta t}} \right)
\]

\[
= \begin{cases} 
(p + q + 1) & \text{if } p > 0 \text{ or } -1 > p, \\
\frac{-q^2}{p-q} & \text{if } p = -1, \\
2(p + 1) & \text{if } p = 0, \\
\frac{p(p-q+1)}{p-q} & \text{if } -1 < p < 0,
\end{cases}
\]

which implies the second part of (3.9).

This completes the proof. ■

Now we can prove the necessity.

Proof of Necessity. If Gini means \(G_{p,q}(a, b)\) are Schur harmonic convex or Schur harmonic concave with respect to \((a, b) \in (0, \infty) \times (0, \infty)\), then

(3.12) \[
\Delta = (a - b) \left( a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} \right) \geq 0 \text{ for all } a, b \in (0, \infty)
\]

or

(3.13) \[
\Delta = (a - b) \left( a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} \right) \leq 0 \text{ for all } a, b \in (0, \infty).
\]

Since \(\Delta\) is symmetric with respect to \(a\) and \(b\), without loss of generality we assume \(a > b\), then \(t = \ln \sqrt{a/b} > 0\). From (3.2) \(\Delta \geq (\leq) 0\) for all \(a, b \in (0, \infty)\) if and only if \(g(t) \geq (\leq) 0\) for all \(t > 0\). It follows that

(3.14) \[
\lim_{t \to 0, t > 0} \frac{g(t)}{t} \geq 0 \text{ and } \lim_{t \to \infty} \max(|A|, |B|, |C|) \frac{2g(t)}{e^{t \max(|A|, |B|, |C|)}} \geq 0
\]

or

(3.15) \[
\lim_{t \to 0, t > 0} \frac{g(t)}{t} \leq 0 \text{ and } \lim_{t \to \infty} \max(|A|, |B|, |C|) \frac{g(t)}{e^{t \max(|A|, |B|, |C|)}} \leq 0.
\]

(1) Firstly, let

\[
\Omega_1 = \{(p, q) : p + q + 1 \geq 0 \text{ and } p \geq q, p \geq 0\},
\]

\[
\Omega_1' = \{(p, q) : p + q + 1 \geq 0 \text{ and } q \geq p, q \geq 0\}.
\]
Then
\[ \Omega_1 \cup \Omega'_1 = \{(p, q) : p + q + 1 \geq 0 \text{ and } \max(p, q) \geq 0\}. \]

Next we prove \((p, q) \in \Omega_1 \cup \Omega'_1\) is the necessary condition for (3.12).

(1.1) **Case 1:** \(p > q\).

From (3.14) we have

(1.1.1) Subcase 1:
\[
\begin{align*}
2(p + q + 1) &\geq 0, \\
p + q + 1 &\geq 0, \\
(p + 1)q &> 0, \\
p(q + 1) &> 0, \\
p &> q.
\end{align*}
\]

That is \((p, q) \in \Omega_{11} = \{(p, q) : p > q > 0\}\).

(1.1.2) Subcase 2:
\[
\begin{align*}
2(p + q + 1) &\geq 0, \\
2p(p + 1)/(p - q) &\geq 0, \\
p &> q = 0.
\end{align*}
\]

That is \((p, q) \in \Omega_{12} = \{(p, q) : p > q = 0\}\).

(1.1.3) Subcase 3:
\[
\begin{align*}
2(p + q + 1) &\geq 0, \\
p(p - q + 1) &\geq 0, \\
p &> -1, \\
q &< 0, \\
p &> q.
\end{align*}
\]

That is \((p, q) \in \Omega_{13} = \{(p, q) : p + q + 1 \geq 0, p \geq 0 > q\}\).

To sum up, \((p, q) \in \Omega_{11} \cup \Omega_{12} \cup \Omega_{13}\).

(1.2) **Case 2:** \(p < q\). Since \(g_{p,q}(t)\) is symmetric with respect to \(p\) and \(q\), we see that if \(G_{p,q}(a, b)\) is Schur harmonic convex then \((p, q) \in \Omega'_{11} \cup \Omega'_{12} \cup \Omega'_{13}\), where
\[
\begin{align*}
\Omega'_{11} &= \{(p, q) : q > p > 0\}, \\
\Omega'_{12} &= \{(p, q) : q > p = 0\}, \\
\Omega'_{13} &= \{(p, q) : p + q + 1 \geq 0, q \geq 0 > p\}.
\end{align*}
\]

(1.3) **Case 3:** \(p = q\). (3.12) holds implies that

(1.3.1) Subcase 1:
\[
\begin{align*}
2(2p + 1) &\geq 0, \\
2p + 1 &\geq 0, \\
p &> 0 \text{ or } p < -1.
\end{align*}
\]
Subcase 2:
\[
\begin{cases}
2(2p + 1) \geq 0, \\
p = 0.
\end{cases}
\]
That is \((p, q) \in \Omega_{10} = \{(p, q) : p = q \geq 0\}.

The above all cases show that if \(G_{p,q}(a, b)\) is Schur harmonic convex then there must be \((p, q) \in \Omega_{11} \cup \Omega_{12} \cup \Omega_{13} \cup \Omega'_{11} \cup \Omega'_{12} \cup \Omega'_{13} \cup \Omega_{10} = \Omega_1 \cup \Omega'_1.\)

(2) Secondly, let
\[
\Omega_2 = \{p + q + 1 \leq 0 \text{ and } \max(p, q) \leq 0\}.
\]
we prove \((p, q) \in \Omega_2\) is the necessary condition for (3.13).

**Case 1**: \(p > q\). (3.13) holds implies that

(2.1.1) Subcase 1:
\[
\begin{cases}
2(p + q + 1) \leq 0, \\
p + q + 1 \leq 0, \\
(p + 1)q > 0, \\
p(q + 1) > 0, \\
p > 1.
\end{cases}
\]
That is \((p, q) \in \Omega_{21} = \{(p, q) : -1 > p > q\}.

(2.1.2) Subcase 2:
\[
\begin{cases}
2(p + q + 1) \leq 0, \\
-2/(p - q) \leq 0, \\
p = -1 > q.
\end{cases}
\]
That is \((p, q) \in \Omega_{22} = \{(p, q) : p = -1 > q\}.

(2.1.3) Subcase 3:
\[
\begin{cases}
2(p + q + 1) \leq 0, \\
p(p - q + 1) \leq 0, \\
p > -1, \\
q < 0, \\
p > q.
\end{cases}
\]
That is \((p, q) \in \Omega_{23} = \{(p, q) : p + q + 1 \leq 0, -1 < p \leq 0, q < 0, p > q\}.

To sum up, \((p, q) \in \Omega_{21} \cup \Omega_{22} \cup \Omega_{23}.\)

**Case 2**: \(p < q\). Since \(g_{p,q}(t)\) is symmetric with respect to \(p\) and \(q\), hence if \(G_{p,q}(a, b)\) is Schur harmonic concave then \((p, q) \in \Omega'_{21} \cup \Omega'_{22} \cup \Omega'_{23},\) where
\[
\begin{align*}
\Omega'_{21} &= \{(p, q) : -1 > q > p > 0\}, & \Omega'_{22} &= \{(p, q) : q = -1 > p\} \\
\Omega'_{23} &= \{(p, q) : p + q + 1 \leq 0, -1 < q \leq 0, p < 0, q > p\}.
\end{align*}
\]

**Case 3**: \(p = q\). (3.13) holds implies that
(2.3.1) Subcase 1:
\[
\begin{aligned}
&\begin{cases}
2(2p+1) \leq 0, \\
2p+1 \leq 0, \\
p > 0 \text{ or } p < -1.
\end{cases}
\Rightarrow p < -1.
\end{aligned}
\]

(2.3.2) Subcase 2:
\[
\begin{aligned}
&\begin{cases}
2(2p+1) \leq 0, \\
p = -1,
\end{cases}
\Rightarrow p = -1.
\end{aligned}
\]

(2.3.3) Subcase 3:
\[
\begin{aligned}
&\begin{cases}
2(2p+1) \leq 0, \\
-1 < p < 0,
\end{cases}
\Rightarrow -1 < p \leq -\frac{1}{2}.
\end{aligned}
\]

That is \((p, q) \in \Omega_{20} = \{(p, q) : p = q \leq -\frac{1}{2}\}\).

The above all cases show that if \(G_{p,q}(a, b)\) is Schur harmonic convex then there must be \((p, q) \in \Omega_{21} \cup \Omega_{22} \cup \Omega_{23} \cup \Omega'_{21} \cup \Omega'_{22} \cup \Omega'_{23} \cup \Omega_{20} = \Omega_2\).

The necessity is proved.

To prove the sufficiency, we need the following lemmas.

**Lemma 5.** \(g(t)\) defined by (3.3) is increasing with \(t\) on \((0, \infty)\) if \((p, q) \in \Omega_1 \cup \Omega'_1\) and decreasing with \(t\) on \((0, \infty)\) if \((p, q) \in \Omega_2\).

**Proof.** (1) we first show \(g'(t) \geq 0\) if \((p, q) \in \Omega_1 \cup \Omega'_1\). By the symmetry of \(p\) and \(q\), we assume \(p \geq q\).

(1.1) In the case of \((p, q) \in \Omega_{11} \cup \Omega_{12}\), that is \(p > q \geq 0\). Then
\[
(p - q)g'(t) = (p - q)\cosh At + (pB - q)\cosh Bt + q(\cosh Bt - \cosh Ct)
+ q(p - q)\cosh Ct
= (p - q)\cosh At + (pB - q)\cosh Bt + 2q\sinh(p - q)t \sinh t
+ q(p - q)\cosh Ct
> 0 \text{ (due to } pB - q = (p + 1)(p - q) > 0)\).
\]

(1.1.2) In the case of \((p, q) \in \Omega_{13}\), that is \(p + q + 1 \geq 0\) and \(p \geq 0, q < 0\).

- If \(p + q + 1 \geq 0\) and \(p \geq 0, -1 < q < 0\), then

\[
(p - q)g'(t) = (p - q)\cosh At + (pB + qp)\cosh Bt - qp(\cosh Bt - \cosh Ct)
- q(p + 1)\cosh Ct
= (p - q)\cosh At + (pB + qp)\cosh Bt - 2qp\sinh(p - q)t \sinh t
- q(p + 1)\cosh Ct > 0 \text{ (due to } pB + qp = p^2 + p \geq 0)\).
\]
In the case of $p > q$
The expression becomes:

$$ (p-q)g'(t) = (p-q)A \cosh At + pB \cosh Bt + qC \cosh Ct $$

Using the identity $\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$, we can simplify:

$$ (p-q)g'(t) = (p-q)A \cosh At + (p-q)A \cosh Bt - 2qp \sinh(p-q)t \sinh t $$

Further simplification:

$$ + 2q(q+1) \sinh(p-q)t \sinh t > 0 $$

Hence $g'(t) \geq 0$ if $(p, q) \in \Omega_{11} \cup \Omega_{12} \cup \Omega_{13}$.

(2.1) In the case of $p = q$. If $(p, q) \in \Omega_{10} = \{(p, q) : p = q \geq 0\}$, then

$$ g'(t) = (2p + 1) \cosh(2p + 1)t + (2p + 1) \cosh t + 2pt \sinh t > 0. $$

These show $g(t)$ defined by (3.3) is increasing with $t$ on $(0, \infty)$ if $(p, q) \in \Omega_1$.

(2) Next we show $g'(t) \leq 0$ if $(p, q) \in \Omega_2$. By the symmetry of $p$ and $q$, we also assume $p \geq q$.

(2.1) In the case of $p > q$.

(2.1.1) In the case of $(p, q) \in \Omega_{21} \cup \Omega_{22}$, that is $-1 \geq p > q$. We have

$$ (p-q)g'(t) = (p-q)A \cosh At + pB \cosh Bt + qC \cosh Ct $$

Using the identity $\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$, we can simplify:

$$ (p-q)g'(t) = (p-q)A \cosh At + (p-q)A \cosh Bt + q(p-q) \cosh Ct $$

Further simplification:

$$ + q(q+1) \cosh Ct > 0 $$

(2.1.2) In the case of $(p, q) \in \Omega_{23} = \{(p, q) : p+q+1 \leq 0, -1 < p \leq 0, q < 0, p > q\}$:

- If $p + q + 1 \leq 0$ and $-1 < p \leq 0, -1 \leq q < 0, p > q$. By (3.17) we have

$$ (p-q)g'(t) = (p-q)A \cosh At + (p-q)A \cosh Bt - 2qp \sinh(p-q)t \sinh t $$

$$ + 2q(q+1) \sinh(p-q)t \sinh t \leq 0. $$

- If $p + q + 1 \leq 0$ and $-1 < p \leq 0, q < -1$. By (3.16) we have

$$ (p-q)g'(t) = (p-q)A \cosh At + p(p+1) \cosh Bt - 2qp \sinh(p-q)t \sinh t $$

$$ - q(q+1) \cosh Ct < 0. $$

(2.2) In the case of $p = q$. If $(p, q) \in \{(p, q) : p = q \leq -\frac{1}{2}\}$, then

$$ g'(t) = (2p + 1) \cosh(2p + 1)t + (2p + 1) \cosh t + 2pt \sinh t < 0, $$

These show $g(t)$ defined by (3.3) is decreasing with $t$ on $(0, \infty)$ if $(p, q) \in \Omega_2$. This completes the proof. □
Now we are in a position to prove the sufficiency.

**Proof of Sufficiency.** (1) By Lemma 5, $g(t)$ defined by (3.3) is increasing with $t$ on $(0, \infty)$ if $(p, q) \in \Omega_1 \cup \Omega'_1$, therefore, $g(t) \geq g(0) = 0$ for all $t \in (0, \infty)$ and then (3.12) holds. From (3.2) and by Lemma 2, $G_{p,q}(a, b)$ is Schur harmonic convex with respect to $(a, b) \in (0, \infty) \times (0, \infty)$.

(2) In the same way, if $(p, q) \in \Omega_2$, $g(t)$ defined by (3.3) is decreasing with $t$ on $(0, \infty)$, therefore, $g(t) \leq g(0) = 0$ for all $t \in (0, \infty)$ and then (3.13) holds, that is $G_{p,q}(a, b)$ is Schur harmonic concave with respect to $(a, b) \in (0, \infty) \times (0, \infty)$.

This completes the proof of sufficiency. \(\blacksquare\)

Thus we complete whole proof of Theorem 2.

**Remark 1.** By the above proofs it is easy to see that

\begin{equation}
\Delta := (a - b) \left( a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} \right) = 0 \text{ for all } a, b \in (0, \infty) \text{ with } a \neq b.
\end{equation}

if and only if $(p, q) \in \{(0, -1), (-1, 0)\}$.

In fact, if $(p, q) \in \{(0, -1), (-1, 0)\}$. It is easy to verify that $g(t) = 0$ for all $t \in (0, \infty)$, which implies (3.18) holds.

On the other hand, if (3.18) holds, it can be divided into three cases.

**Case 1:** $p > q$. From (3.7) we see that

\begin{align*}
p + q + 1 &> 1 \text{ or } < -1 & \text{ if } p > q > 0 \text{ or } -1 > p > q, \\
-\frac{q^2}{p - q} &< 0 & \text{ if } p = -1 > q, \\
\frac{2p(p + 1)}{(p - q)} &> 0 & \text{ if } p > q = 0, \\
\frac{p(p - q + 1)}{(p - q)} &< 0 \text{ or } = 0 \text{ or } > 0 & \text{ if } p > -1, q < 0, p > q.
\end{align*}

It follows from (3.18) together with (3.4) and 3.7 that

\begin{align*}
2(p + q + 1) &= 0, \\
p(p - q + 1)/(p - q) &= 0 \text{ if } p > -1, q < 0, p > q.
\end{align*}

Solving the equations yields $(p, q) = (0, -1)$.

**Case 2:** $p < q$. By the symmetry of $p$ and $q$ we also have $(p, q) = (-1, 0)$.

**Case 3:** $p = q$. we have

\begin{align*}
2(p + q + 1) &= 0, \\
2p + 1 &= 0 \text{ if } p > 0 \text{ or } p < -1.
\end{align*}

Obviously, this equations have not any solutions.

Therefore, if (3.18) is true, then $(p, q) \in \{(0, -1), (-1, 0)\}$.

**References**


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