Iteration Product of Nörlund Methods of Double Sequences in Non-Archimedean Fields

V. Srinivasan
Department of Mathematics
Faculty of Engineering and Technology
SRM University, Kattankulathur, 603203, India
drvsrinivas.5@gmail.com

R. Deepa
Department of Mathematics
SRM University, India

Abstract

The aim of the paper is to define the iteration product of Nörlund methods of double sequences in a complete, non-trivially valued, non-archimedean field and prove a few inclusion theorems on the iteration product of Nörlund methods of double sequences in such fields.

Mathematics Subject Classification: 40G, 46S

Keywords: Non-archimedean fields, double sequences, generalized Nörlund methods, bi-regular Nörlund methods

1 Introduction

For analysis in the classical case a general reference is [2] while for analysis in non-archimedean fields a general reference is [1].

If $A = (a_{m,n,k,l})$, $a_{m,n,k,l} \in K$, $m,n,k,l = 0,1,2,\ldots$ is an infinite matrix, the $A$-transform $Ax = \{(Ax)_{m,n}\}$ of the double sequence $x = \{x_{k,l}\}$, $x_{k,l} \in K$, $k,l = 0,1,2,\ldots$ is defined by
\{(Ax)_{m,n}\} = y_{m,n} = \sum_{k=0, l=0}^{\infty, \infty} a_{m,n,k,l} x_{k,l}, \ m, n = 0, 1, 2, \ldots, \text{assuming that the series on the right converge.}

In the sequel, the following definitions are needed.

**Definition 1.1.** Let \( \{x_{m,n}\} \) be a double sequence in \( K \) and \( x \in K \). We say that

\[
\lim_{m+n \to \infty} x_{m,n} = x, \text{ if for each } \epsilon > 0 \text{ the set } \{(m,n) \in \mathbb{N}^2 : |x - x_{m,n}| \geq \epsilon\} \text{ is finite. In such a case we say that } x \text{ is the limit of } \{x_{m,n}\}.
\]

**Definition 1.2.** Let \( \{x_{m,n}\} \) be a double sequence in \( K \) and \( s \in K \). We say that

\[
s = \sum_{m=0, n=0}^{\infty, \infty} x_{m,n}
\]

if

\[
s = \lim_{m+n \to \infty} s_{m,n}
\]

where

\[
s_{m,n} = \sum_{k=0, l=0}^{m, n} x_{k,l}, \ m, n = 0, 1, 2, \ldots.
\]

**Remark 1.1.** If \( \lim_{m+n \to \infty} x_{m,n} = x \), then the sequence \( \{x_{m,n}\} \) is automatically bounded.

It is easy to prove the following results.

**Lemma 1.1.** \( \lim_{m+n \to \infty} x_{m,n} = x \) if and only if

(i) \( \lim_{n \to \infty} x_{m,n} = x, \ m = 0, 1, 2, \ldots, \)

(ii) \( \lim_{m \to \infty} x_{m,n} = x, \ n = 0, 1, 2, \ldots \) and

(iii) for each \( \epsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that \( |x - x_{m,n}| < \epsilon, \) for all \( m, n \geq N \) which we write as \( \lim_{m,n \to \infty} x_{m,n} = x. \)

**Lemma 1.2.** \( \lim_{m+n \to \infty} s_{m,n} \) exists if and only if \( \lim_{m+n \to \infty} x_{m,n} = 0. \)

**Definition 1.3.** If \( \lim_{m+n \to \infty} y_{m,n} = s \) whenever \( \lim_{m+n \to \infty} x_{m,n} = s, \) we say that \( A \) is regular.
The following result is well known (see [10]).

**Theorem 1.1.** In order that whenever a sequence \( \{x_{m,n}\} \) has a limit \( x \),
\[
\sum_{k=0, l=0}^{\infty, \infty} a_{m,n,k,l}x_{k,l}
\]
shall converge and
\[
\lim_{m+n \to \infty} \sum_{k=0, l=0}^{\infty, \infty} a_{m,n,k,l}x_{k,l} = x
\]
i.e., for \( A = (a_{m,n,k,l}) \) to be regular it is necessary and sufficient that

\[(i) \quad \lim_{m+n \to \infty} a_{m,n,k,l} = 0, \quad k, l = 0, 1, 2, \ldots,

(ii) \quad \sum_{k=0, l=0}^{\infty, \infty} a_{m,n,k,l} = 1; \text{ for every fixed } k \text{ and } l,

(iii) \quad \lim_{m+n \to \infty} \sup_{k \geq 1} |a_{m,n,k,l}| = 0, \quad l = 0, 1, 2, \ldots,

(iv) \quad \lim_{m+n \to \infty} \sup_{l \geq 1} |a_{m,n,k,l}| = 0, \quad k = 0, 1, 2, \ldots \text{ and}

(v) \quad \sup_{m,n,k,l} |a_{m,n,k,l}| < \infty.

**Definition 1.4.** The Nörlund method of summability in \( K \), denoted by \((N, p_{m,n})\)
is defined by the infinite matrix \((a_{m,n,k,l})\) where

\[
a_{m,n,k,l} = \begin{cases} 
\frac{p_{m-k,n-l}}{P_{m,n}} & k \leq m, l \leq n \\
0 & k > m, l > n
\end{cases}
\]

where \( p_{0,0} \neq 0, |p_{0,j}| > |p_{i,j}|, i, j = 1, 2, \ldots, \) and \( P_{m,n} = \sum_{k=0, l=0}^{m,n} p_{k,l} \),

\[ m, n = 0, 1, 2, \ldots \]

**Theorem 1.2 (see [10], Theorem 3).** The necessary and sufficient conditions
for the regularity of the Nörlund means \((N, p_{m,n})\) are

\[
\lim_{m+n \to \infty} \sup_{0 \leq k \leq n} |p_{m-k,n-l}| = 0, \quad 0 \leq k \leq m;
\]

\[
\lim_{m+n \to \infty} \sup_{0 \leq l \leq n} |p_{m-k,n-l}| = 0, \quad 0 \leq l \leq n.
\]

We will denote \((N, p_{m,n})\) as \( N^p_{m,n} \), which we write as \((N, p')\). Similarly, we
can denote \((N, q_{m,n})\) and \((N, t_{m,n})\) as \( N^q_{m,n} \) and \( N^t_{m,n} \), which we write as \((N, q')\)
and \((N, t')\) respectively.

Let \((N, p')\) and \((N, q')\) be two given regular Nörlund methods.
For \( m, n = 0, 1, 2, \ldots \), we define
\[
t_{m,n} = \sum_{k=0, l=0}^{m,n} p_{k,l} q_{m-k,n-l},
\]
such that \( p_{m,n} = -p_{n,m} \) and \( q_{m,n} = -q_{n,m} \).

i.e., \( t_{m,n} = p_{0,0} q_{m,n} + p_{1,1} q_{m-1,n-1} + \cdots + p_{m,n} q_{0,0} \), \( m, n = 0, 1, 2, \ldots \).

Let
\[
p'(x, y) = \sum_{m,n=0}^{\infty, \infty} p_{m,n} x^m y^n,
\]
\[
q'(x, y) = \sum_{m,n=0}^{\infty, \infty} q_{m,n} x^m y^n \text{ and}
\]
\[
t'(x, y) = \sum_{m,n=0}^{\infty, \infty} t_{m,n} x^m y^n.
\]

So
\[
t'(x, y) = p'(x, y) q'(x, y).
\] (1)

Also,
\[
T_{m,n} = t_{0,0} + t_{1,1} + \cdots + t_{m,n} \neq 0, \quad m, n = 0, 1, 2, \ldots.
\]

For,
\[
p_{0,0} q_{0,0} \neq 0; |T_{m,n}| = |p_{0,0} Q_{m,n} + p_{1,1} Q_{m-1,n-1} + \cdots + p_{m,n} Q_{0,0}|
\]
\[
= |p_{0,0} q_{0,0}| \neq 0, \quad \text{since} \ |Q_{m,n}| = |q_{0,0}|, \quad m, n = 0, 1, 2, \ldots.
\]

We now define the double sequences \( \{f_{m,n}\} \), \( \{g_{m,n}\} \) and \( \{h_{m,n}\} \) as follows.

Definition 1.5.

\[
(i) \quad \frac{1}{t'(x, y)} = \sum_{m=0, n=0}^{\infty, \infty} f_{m,n} x^m y^n, f_{-1,-1} = 0,
\]
\[
(ii) \quad \frac{q'(x, y)}{p'(x, y)} = \sum_{m=0, n=0}^{\infty, \infty} g_{m,n} x^m y^n, g_{-1,-1} = 0 \text{ and}
\]
\[
(iii) \quad \frac{q'(x, y)}{p'(x, y) t'(x, y)} = \sum_{m=0, n=0}^{\infty, \infty} h_{m,n} x^m y^n, h_{-1,-1} = 0.
\] (2)

We shall denote the \((N, p')\) transform of the double sequence \( \{s_{m,n}\} \) by
\[
N^p_{m,n} = \frac{1}{P_{m,n}} \sum_{\alpha=0, \beta=0}^{m,n} p_{m-\alpha, n-\beta} s_{\alpha, \beta}.
\]

Similarly, the \((N, p')\) transform of the \((N, q')\) transform of the double sequence is denoted by \((N^{p', q})\) and the corresponding summability method by \((N, p')(N, q')\).
P.N. Natarajan and V. Srinivasan (see [11]) proved a few theorems for Nörlund methods based on this iteration product in $K$.

**Theorem 1.3.** Let $(N, p')$, $(N, q')$ and $(N, t')$ be regular Nörlund methods. Then

$$(N, t')(N, p') \subseteq (N, t')(N, q').$$

(3)

if and only if

$$\sup_{m, n, \gamma, \mu} |\lambda_{m,n,\gamma,\mu}| < \infty,$$

(4)

$$\lim_{m+n \to \infty} \lambda_{m,n,\gamma,\mu} = 0, \ \gamma, \mu = 0, 1, 2, \ldots$$

(5)

and

$$\lim_{m+n \to \infty} \sup_{\gamma \geq 0} |\lambda_{m,n,\gamma,\mu}| = 0, \ \mu = 0, 1, 2, \ldots,$$

(6)

where

$$\lambda_{m,n,\gamma,\mu} = \begin{cases} \frac{T_{\gamma,\mu}}{T_{m,n}} \sum_{\alpha=\gamma, \beta=\mu} t_{m-n, \alpha, \beta} Q_{\alpha, \beta} \sum_{\eta=\gamma, \theta=\mu} g_{\alpha-\eta, \beta-\theta} f_{\eta-\gamma, \theta-\mu} P_{n,\theta}, & \gamma \leq m, \mu \leq n \\ 0, & \gamma > m, \mu > n. \end{cases}$$

(7)
Proof.

\[
\sum_{\alpha=0,\beta=0}^{m,n} \frac{g_{m-\alpha,n-\beta}}{Q_{m,n}} \alpha,\beta \sum_{\eta=0,\theta=0}^{\alpha,\beta} p_{\alpha-\eta,\beta-\theta} \gamma \eta,\theta
\]

\[
= \sum_{\alpha=0,\beta=0}^{m,n} \left(\frac{g_{m-\alpha,n-\beta}}{Q_{m,n}} \right) \alpha,\beta \sum_{\eta=0,\theta=0}^{\alpha,\beta} p_{\alpha-\eta,\beta-\theta} \gamma \eta,\theta
\]

\[
= \sum_{\alpha=0,\beta=0}^{m,n} \frac{g_{m-\alpha,n-\beta}}{Q_{m,n}} \left(\sum_{\eta=0,\theta=0}^{\alpha,\beta} p_{\alpha-\eta,\beta-\theta} \gamma \eta,\theta\right)
\]

\[
= \frac{1}{Q_{m,n}} \left[ g_{m,n} p_{0,0} s_{0,0} + g_{m-1,n-1} \left(\sum_{\eta=0,\theta=0}^{1,1} p_{1-\eta,1-\theta} \gamma \eta,\theta\right) \right.
\]

\[
+ g_{m-2,n-2} \left(\sum_{\eta=0,\theta=0}^{2,2} p_{2-\eta,2-\theta} \gamma \eta,\theta\right) + \cdots
\]

\[
+ g_{0,0} \left(\sum_{\eta=0,\theta=0}^{m,n} p_{m-\eta,n-\theta} \gamma \eta,\theta\right) \right]
\]

\[
= \frac{1}{Q_{m,n}} \left[ g_{m,n} p_{0,0} s_{0,0} + g_{m-1,n-1} (p_{1,1} s_{0,0} + p_{0,0} s_{1,1}) \right.
\]

\[
+ g_{m-2,n-2} (p_{2,2} s_{0,0} + p_{1,1} s_{1,1} + p_{0,0} s_{2,2}) + \cdots
\]

\[
+ g_{0,0} (p_{m,n} s_{0,0} + p_{m-1,n-1} s_{1,1} + \cdots + p_{0,0} s_{m,n}) \right]
\]

\[
= \frac{1}{Q_{m,n}} \left[ (g_{m,n} p_{0,0} + g_{m-1,n-1} p_{1,1} + g_{m-2,n-2} p_{2,2} + \cdots + g_{0,0} p_{m,n}) s_{0,0} \right.
\]

\[
+ (g_{m-1,n-1} p_{0,0} + g_{m-2,n-2} p_{1,1} + \cdots + g_{0,0} p_{m-1,n-1}) s_{1,1} + \cdots
\]

\[
+ (g_{0,0} p_{0,0}) s_{m,n} \right]
\]

\[
= \frac{1}{Q_{m,n}} \left[ q_{m,n} s_{0,0} + \cdots + q_{0,0} s_{m,n} \right], \text{ using (ii) of (2)}
\]

\[
= \frac{1}{Q_{m,n}} \sum_{\alpha=0,\beta=0}^{m,n} q_{m-\alpha,n-\beta} \gamma \alpha,\beta
\]

\[
= N_{m,n}^q.
\]

Thus

\[
N_{m,n}^q = \sum_{\alpha=0,\beta=0}^{m,n} \left(\frac{g_{m-\alpha,n-\beta}}{Q_{m,n}} \right) P_{\alpha,\beta} N_{\alpha,\beta}^p.
\]
Consider
\[
\sum_{\alpha=0,\beta=0}^{m,n} f_{m-a,n-\beta} T_{a,\beta} N_{a,\beta}^{t,p}
\]
\[
= \sum_{\alpha=0,\beta=0}^{m,n} f_{m-a,n-\beta} T_{a,\beta} \left( \frac{1}{T_{a,\beta}} \sum_{\eta=0,\theta=0}^{\alpha,\beta} t_{a-\eta,\beta-\theta} N_{\eta,\theta}^{p} \right),
\]
\[
= \left[ f_{m,n} t_{0,0} N_{0,0}^{p} + f_{m-1,n-1} \left( \sum_{\eta=0,\theta=0}^{1,1} t_{1-\eta,1-\theta} N_{\eta,\theta}^{p} \right) \right. \\
+ \left. f_{m-2,n-2} \left( \sum_{\eta=0,\theta=0}^{2,2} t_{2-\eta,2-\theta} N_{\eta,\theta}^{p} \right) + \cdots + f_{0,0} \left( \sum_{\eta=0,\theta=0}^{m,n} t_{m-\eta,n-\theta} N_{\eta,\theta}^{p} \right) \right]
\]
\[
= \left[ f_{m,n} t_{0,0} N_{0,0}^{p} + f_{m-1,n-1} \left( t_{1,1} N_{0,0}^{p} + t_{0,0} N_{1,1}^{p} \right) \right. \\
+ \left. f_{m-2,n-2} \left( t_{2,2} N_{0,0}^{p} + t_{1,1} N_{1,1}^{p} + t_{0,0} N_{2,2}^{p} \right) + \cdots \right. \\
+ \left. f_{0,0} \left[ t_{m,n} N_{0,0}^{p} + t_{m-1,n-1} N_{1,1}^{p} + \cdots + t_{0,0} N_{m,n}^{p} \right] \right]
\]
\[
= \left[ (f_{m,n} t_{0,0} + f_{m-1,n-1} t_{1,1} + \cdots + f_{0,0} t_{m,n}) N_{0,0}^{p} \right. \\
+ \left. (f_{m-1,n-1} t_{0,0} + f_{m-2,n-2} t_{1,1} + \cdots + f_{0,0} t_{m-1,n-1}) N_{1,1}^{p} + \cdots \right. \\
+ \left. (f_{0,0} t_{0,0} N_{m,n}^{p}) \right]
\]
\[
= N_{m,n}^{p}, \text{ using (i) of (2)}. 
\]

Thus
\[
N_{m,n}^{p} = \sum_{\alpha=0,\beta=0}^{m,n} f_{m-a,n-\beta} T_{a,\beta} N_{a,\beta}^{t,p}. \quad (9)
\]

Now, we consider
\[
N_{m,n}^{t,q} = \frac{1}{T_{m,n}} \sum_{\alpha=0,\beta=0}^{m,n} t_{m-a,n-\beta} N_{a,\beta}^{q}
\]
\[
= \frac{1}{T_{m,n}} \sum_{\alpha=0,\beta=0}^{m,n} t_{m-a,n-\beta} \frac{1}{Q_{a,\beta}} \sum_{\eta=0,\theta=0}^{\alpha,\beta} g_{a-\eta,\beta-\theta} P_{\eta,\theta} N_{a,\beta}^{p}
\]
\[
= \frac{1}{T_{m,n}} \sum_{\alpha=0,\beta=0}^{m,n} \left( \frac{t_{m-a,n-\beta}}{Q_{a,\beta}} \right) \sum_{\eta=0,\theta=0}^{\alpha,\beta} g_{a-\eta,\beta-\theta} P_{\eta,\theta} \sum_{\gamma=0,\mu=0}^{\eta,\theta} f_{\eta-\gamma,\mu} T_{\gamma,\mu} N_{\gamma,\mu}^{t,p}
\]
\[
= \sum_{\gamma=0,\mu=0}^{\eta,\theta} \left\{ \frac{T_{\gamma,\mu}}{T_{m,n}} \sum_{\alpha=0,\beta=0}^{m,n} \left( \frac{t_{m-a,n-\beta}}{Q_{a,\beta}} \right) \sum_{\eta=0,\theta=0}^{\alpha,\beta} g_{a-\eta,\beta-\theta} P_{\eta,\theta} \right\} N_{\gamma,\mu}^{t,p}
\]
\[
= \sum_{\gamma=0,\mu=0}^{\eta,\theta} \lambda_{m,n,\gamma,\mu} N_{\gamma,\mu}^{t,p}, \text{ using (7)}. \quad (10)
\]
From (10) we get \((N, t')(N, p') \subseteq (N, t')(N, q')\) if and only if \(\lambda_{m,n,\gamma,\mu}\) is regular.

In view of Theorem 1.1, \((\lambda_{m,n,\gamma,\mu})\) is regular if and only if (4), (5) and (6) hold, since

\[
\sum_{\gamma=0,\mu=0}^{\infty,\infty} \lambda_{m,n,\gamma,\mu} = \sum_{\gamma=0,\mu=0}^{m,n} \lambda_{m,n,\gamma,\mu} = 1, \quad m, n = 0, 1, 2, \ldots,
\]

and so

\[
\lim_{m+n \to \infty} \sum_{\gamma=0,\mu=0}^{\infty,\infty} \lambda_{m,n,\gamma,\mu} = 1
\]

always holds, thus completing the proof of the theorem.

\[\square\]

**Theorem 1.4.** Let the methods \((N, p')\), \((N, t')\) and \((N, k')\) be regular Nörlund methods and

\[
\lim_{m+n \to \infty} f_{m,n} = 0.
\]

Then

\((N, t')(N, p') \subseteq (N, t')(N, q')\).

**Proof.** For \(\gamma, \mu = 0, 1, 2, \ldots\), we have

\[
|\lambda_{m,n,\gamma,\mu}| = \left| \frac{T_{\gamma,\mu}}{T_{m,n}} \sum_{\alpha=\gamma,\beta=\mu}^{m,n} \frac{t_{m-a,n-\beta}}{Q_{a,\beta}} \left( \sum_{\eta=\gamma,\theta=\mu}^{a,\beta} g_{a-\eta,\beta-\theta} f_{\eta-\gamma,\theta-\mu} P_{\eta,\theta} \right) \right|
\]

\[
\leq \max \left[ \left| \frac{t_{m-a,n-\beta}}{Q_{a,\beta}} g_{0,0} f_{0,0} P_{\gamma,\mu} \right| \cdots \left| \frac{t_{0,0}}{Q_{m,n}} \sum_{\eta=\gamma,\theta=\mu}^{a,\beta} g_{a-\eta,\beta-\theta} f_{\eta-\gamma,\theta-\mu} \right| \right]
\]

\[
\leq \max \left[ |t_{m-\gamma,n-\mu}| |f_{0,0} g_{0,0}|, |t_{m-\gamma-1,n-\mu-1}| |g_{1,1} f_{0,0} + f_{1,1} g_{0,0}| \cdots \right.
\]

\[
|t_{0,0}| g_{m-\gamma,n-\mu} f_{0,0} + \cdots + g_{0,0} f_{m-\gamma,n-\mu} |]
\]

\[
\leq \max \left[ |g_{m-\gamma,n-\mu}|, |g_{m-\gamma-1,n-\mu-1}| |f_{1,1}| \cdots |g_{0,0}| \right]
\]

\[
\to 0 \quad (m + n \to \infty),
\]

since \(\lim_{m+n \to \infty} f_{m,n} = 0\) and also using the fact that \(\lim_{m+n \to \infty} g_{m,n} = 0\), \((N, k')\) being a regular method. i.e., \(\lim_{m+n \to \infty} \lambda_{m,n,\gamma,\mu} = 0, \gamma, \mu = 0, 1, 2, \ldots\) It is clear that (4) and (6) hold. In view of Theorem 1.3, the proof is complete. \(\square\)
Lemma 1.3. Let \((N, p')\), \((N, t')\) be regular Nörlund methods. Then \((N, t')(N, p')\) is also regular if and only if

\[
\lim_{m+n \to \infty} c_{m,n,k,l} = 0, \quad k, l = 0, 1, 2, \ldots,
\]

where

\[
c_{m,n,k,l} = \frac{1}{T_{m,n}} \left[ \sum_{\alpha=0,\beta=0}^{m,n} \frac{p_{\alpha,\beta} t_{k-\alpha,l-\beta}}{P_{m-(k-\alpha),n-(l-\beta)}} \right].
\]

Proof. The \((N, p')\) transform of \(\{s_{k,l}\}\) is

\[
u_{m,n} = \frac{1}{P_{m,n}} \sum_{\alpha=0,\beta=0}^{m,n} p_{\alpha,\beta} s_{m-\alpha,n-\beta}
\]

Then \((N, t')(N, p')\) transform of \(\{s_{k,l}\}\) is

\[
= \frac{1}{T_{m,n}} (t_{0,0} u_{m,n} + t_{1,1} u_{m-1,n-1} + \cdots + t_{m,n} u_{0,0})
\]

\[
= \frac{1}{T_{m,n}} \left[ t_{0,0} \left( \frac{1}{P_{m,n}} \sum_{\alpha=0,\beta=0}^{m,n} p_{\alpha,\beta} s_{m-\alpha,n-\beta} \right) + t_{1,1} \left( \frac{m,n-1}{P_{m-1,n-1}} \sum_{\alpha=0,\beta=0}^{m,n} p_{\alpha,\beta} s_{m-\alpha-1,n-\beta-1} \right) + \cdots \right]
\]

\[
= \frac{1}{T_{m,n}} \left[ t_{1,1} \left( \frac{P_{0,0} s_{m-1,n-1} + P_{1,1} s_{1,1} + \cdots + P_{m,n} s_{0,0}}{P_{m-1,n-1}} \right) + t_{m,n} \left( \frac{P_{0,0} s_{0,0}}{P_{0,0}} \right) \right]
\]

\[
= \frac{1}{T_{m,n}} \left[ t_{0,0} \frac{P_{0,0} s_{m,n} + P_{1,1} s_{m-1,n-1} + \cdots + P_{m,n} s_{0,0}}{P_{m,n}} \right]
\]

\[
+ \frac{t_{1,1}}{P_{m-1,n-1}} \left( \frac{P_{0,0} s_{m-1,n-1} + P_{1,1} s_{m-2,n-2} + \cdots + P_{m-1,n-1} s_{0,0}}{P_{1,1}} + \frac{t_{m,n}}{P_{0,0}} \right)
\]

\[
= \frac{1}{T_{m,n}} \left[ t_{0,0} \frac{P_{0,0} s_{m,n} + t_{0,0} \frac{P_{1,1} s_{m,n}}{P_{m,n}} + t_{1,1} \frac{P_{0,0} s_{m,n}}{P_{m,n}}}{P_{m,n}} + \cdots \right]
\]

\[
+ \left( \frac{t_{0,0} P_{m-1,n-1} + \cdots + t_{m,n} P_{0,0}}{P_{1,1}} \right) s_{1,1}
\]

\[
+ \left( \frac{t_{0,0} P_{m,n} + \cdots + t_{m,n} P_{0,0}}{P_{0,0}} \right) s_{0,0}. \]
Thus \((N, t')(N, p')\) is given by the infinite matrix \((c_{m,n,k,l})\), where

\[
c_{m,n,k,l} = \begin{cases} 
\frac{1}{T_{m,n}} \left[ \sum_{\alpha=0, \beta=0}^{k,l} \frac{p_{\alpha,\beta} t_{k-\alpha,l-\beta}}{P_{m-(k-\alpha), n-(l-\beta)}} \right], & k \leq m, l \leq n \\
0, & k > m, l > n.
\end{cases}
\]

It is evident that

\[
\sup_{m,n,k,l} |c_{m,n,k,l}| < \infty,
\]

since \(\{p_{m,n}\}\) is bounded and \(\{t_{m,n}\}\) is bounded, using remark, we have \(\{c_{m,n,k,l}\}\) is also bounded.

i.e., \(\lim_{m+n \to \infty} \sup_{k \geq 0} |c_{m,n,k,l}| = 0, \ l = 0, 1, 2, \ldots\)

Similarly,

\[
\lim_{m+n \to \infty} \sup_{l \geq 0} |c_{m,n,k,l}| = 0, \ k = 0, 1, 2, \ldots
\]

Also,

\[
\sum_{k=0,l=0}^{\infty,\infty} c_{m,n,k,l} = \sum_{k=0,l=0}^{m,n} c_{m,n,k,l}
\]

\[
= \frac{1}{T_{m,n}} \left[ \frac{p_{0,0} t_{0,0}}{P_{m,n}} + \left( \frac{p_{0,0} t_{1,1}}{P_{m-1,n-1}} + \frac{p_{1,1} t_{0,0}}{P_{m,n}} \right) + \left( \frac{p_{0,0} t_{2,2}}{P_{m-2,n-2}} + \frac{p_{1,1} t_{1,1}}{P_{m-1,n-1}} + \frac{p_{2,2} t_{0,0}}{P_{m,n}} \right) \right] + \cdots + \left( \frac{p_{0,0} t_{m,n}}{P_{0,0}} + \frac{p_{1,1} t_{m-1,n-1}}{P_{1,1}} + \cdots + \frac{p_{m,n} t_{0,0}}{P_{m,n}} \right)
\]

\[
= \frac{1}{T_{m,n}} \left[ \frac{t_{0,0}}{P_{m,n}} (p_{m,n}) + \frac{t_{1,1}}{P_{m-1,n-1}} (p_{m-1,n-1}) + \cdots + \frac{t_{m,n}}{P_{0,0}} (p_{0,0}) \right]
\]

\[
= \frac{1}{T_{m,n}} \left[ t_{0,0} + t_{1,1} + \cdots + t_{m,n} \right]
\]

\[
= \frac{T_{m,n}}{T_{m,n}}
\]

\[
= 1, \ m, n = 0, 1, 2, \ldots
\]

so that

\[
\lim_{m+n \to \infty} \sum_{k=0,l=0}^{\infty,\infty} c_{m,n,k,l} = 1.
\]

In view of Theorem 1.1, the proof of the lemma is complete. \(\square\)
Theorem 1.5. Let \((N, p'), (N, q')\) and \((N, t')\) be regular Nörlund methods. Let (12) hold and

\[
\lim_{m+n\to\infty} d_{m,n,k,l} = 0, \quad k, l = 0, 1, 2, \ldots, \tag{13}
\]

where

\[
d_{m,n,k,l} = \begin{cases} 
\frac{1}{T_{m,n}} \left[ \sum_{\alpha=0,\beta=0}^{k,l} q_{\alpha,\beta} t_{k-\alpha,l-\beta} \right], & k \leq m, l \leq n \\
0, & k > m, l > n.
\end{cases}
\]

If \((N, p') \subseteq (N, q')\), then

\[
(N, t')(N, p') \subseteq (N, t')(N, q'). \tag{14}
\]

Proof. Consider

\[
v'_m,n = \sum_{\alpha=0,\beta=0}^{m,n} t_{m-n-\beta} p_{\alpha,\beta} \quad \text{and} \quad v''_m,n = \sum_{\alpha=0,\beta=0}^{m,n} t_{m-n-\beta} q_{\alpha,\beta}, \quad m, n = 0, 1, 2, \ldots. \tag{15}
\]

Since \((N, p'), (N, q'), (N, t')\) are regular methods, \((N, t')(N, p')\) and \((N, t')(N, q')\) are also regular in view of Lemma 1.3, (12) and (13). To prove (14), it is enough to prove that \((N, v') \subseteq (N, v'')\).

Let \(v'(x, y) = \sum_{m=0, n=0}^{\infty} v'_m,n x^m y^n\) and \(v''(x, y) = \sum_{m=0, n=0}^{\infty} v''_m,n x^m y^n\).

From (15) we have

\[
p'(x, y) = \frac{v'(x, y)}{v'(x, y)} \tag{16}
\]

and

\[
q'(x, y) = \frac{v''(x, y)}{v'(x, y)} \tag{17}
\]

Since \((N, p') \subseteq (N, q')\), \(\lim_{m+n\to\infty} g_{m,n} = 0\), where \(g(x, y) = \frac{q'(x,y)}{p'(x,y)}\).

Using (16) and (17), we have

\[
g(x, y) = \frac{q'(x,y)}{p'(x,y)} = \frac{v''(x, y)}{v'(x, y)},
\]

where \(\lim_{m+n\to\infty} g_{m,n} = 0\). Again from Theorem 5 (see [11]), we have

\[
(N, v') \subseteq (N, v''),
\]

which completes the proof of the theorem. \qed
References


Received: August, 2010