A Common Fixed Point Theorem for $\Phi$-Weakly Commuting Mappings in Metric Spaces

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Abstract

In this paper we introduce the concept of $\Phi$-weakly commuting maps and show that the result of S.L. Singh, A. Hematulin and R. Pant [7] holds even if conditions ($A_1$) and ($A_3$) of Theorem 2.3 are dropped under certain restrictions (Theorem 2.5). We provide example and generalization to the same with some modifications.

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1 Introduction


Throughout this paper $R^+$ is the set of non negative real numbers and $Y$ is an arbitrary non empty set. For $T, f : Y \to X$, let $C(T, f)$ denote the set of coincidence points of $T$ and $f$ i.e. $C(T, f) = \{z \in Y : Tz = fz\}$. 
We start with

**Definition 1.1**: [5] Let $\Phi$ denote the class of all functions $\varphi : R^+ \rightarrow R^+$ satisfying:

for any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < t < \delta$ implies $\varphi(t) \leq \epsilon$.

**Definition 1.2**: [2] Let $S$, $T$ and $f$ be self maps on $Y$ with values in a metric space $(X,d)$. The pair $(S,T)$ is asymptotically regular with respect to $f$ at $x_0 \in Y$ if there exists a sequence $\{x_n\}$ in $Y$ such that $fx_{2n+1} = Sx_{2n}, fx_{2n+2} = Tx_{2n+1}, n \in \{0,1,2,\ldots\}$ and $\lim_{n \to \infty} d(fx_n, fx_{n+1}) = 0$.

$(T,f)$ is asymptotically regular at $x_0 \in X$ if there exists a sequence $\{x_n\}$ such that $fx_{2n+1} = fx_{2n}, fx_{2n+2} = Tx_{2n+1}, n \in \{0,1,2,\ldots\}$ and $\lim_{n \to \infty} d(fx_n, fx_{n+1}) = 0$.

If $Y = X$ and $S = T$, then we get the definition of asymptotic regularity of $T$ with respect to $f$ due to Rhoades et.al [6].

**Definition 1.3**: [4] Let $(X,d)$ be a metric space and $T, f : X \to X$. Then they are said to be $R$-weakly commuting if there exists a positive real number $R$ such that $d(Tfx, fTx) \leq R(d(Tx, fx))$ for all $x \in X$.

**Definition 1.4**: [7] Let $T, f : X \to X$. Then the pair $(T, f)$ is $(IT)$-commuting at $z \in X$ if $Tfz = fTz$. They are $(IT)$-commuting on $X$ (also called weakly compatible, by Jungck and Rhoades [2]) if $Tfz = fTz$ for all $z \in X$ such that $Tz = fz$.

The following theorem is proved in [5].

**Theorem 1.5**: [5] Let $T$ be a continuous and asymptotically regular self map on a complete metric space $(X,d)$ satisfying the following conditions:

$(P_1)$ $d(Tx, Ty) \leq \varphi(D(x,y))$ for all $x, y \in X$

$(P_2)$ $d(Tx, Ty) < D(x,y)$ for all distinct $x, y \in X$ where

$D(x,y) = d(x,y) + \gamma[d(x,Tx) + d(y, Ty)], \gamma \geq 0$ and $\varphi \in \Phi$.

Then $T$ has a unique fixed point.

Moreover if $D(x,y) = d(x,y) + d(x,Tx) + d(y, Ty)$ (i.e. $\gamma = 1$) and $\varphi$ is continuous and satisfies $\varphi(t) > t$ for all $t > 0$, then continuity of $T$ can be dropped.

The following theorem is claimed to be a generalization of Theorem 1.5 in [7].

**Theorem 1.6** (S.L. Singh et. al [7], Theorem 2.3): Let $T$ and $f$ be self maps on a complete metric space $(X,d)$ such that

$(A_1)$ $T(X) \subseteq f(X)$

$(A_2)$ $d(Tx, Ty) \leq \varphi(g(x,y))$ for all $x, y \in X$ where $g(x,y) = d(fx, fy) + \gamma[d(fx, Tx) + d(fy, Ty)],$ for some $\gamma \geq 0$ and $\varphi \in \Phi$ is continuous.

$(A_3)$ $d(Tx, Ty) < g(x,y)$ for all distinct $x, y \in X$

$(A_4)$ $(T, f)$ is asymptotically regular at $x_0 \in X$.

If $T$ is continuous then $T$ has a fixed point provided that $T$ and $f$ are $R$-weakly commuting. Further if $f$ is continuous and $\gamma = 1$, then $T$ and $f$ have a unique
common fixed point provided that T and f are R-weakly commuting.

The above theorem is not valid if \((A_1)\) is dropped, in view of the following example. However, in proving the theorem, the authors used \((A_1)\) to construct a sequence \(\{x_n\}\) and hence \(\{y_n\}\), beginning with a point \(x_0 \in X\). Using \((A_4)\) the authors claimed that \(\lim_{n \to \infty} d(y_n, y_{n+1}) = 0\) for the above sequence \(\{y_n\}\), which need not be true.

Further, the authors assumed tacitly that \(\varphi(t) < t \text{ if } t > 0\), to arrive at a contradiction, but for the \(\varphi\) under consideration in the theorem this may not hold. This is evident from the following example.

**Example 1.7:** Let \(X = [0, 1]\) with usual metric \(d\). Let \(T, f : X \to X\) be defined by \(Tx = 1 - x^2\) and \(fx = 1 \forall x \in X\). Let \(\varphi : R^+ \to R^+\) be defined by

\[
\varphi(x) = \begin{cases} 
  x^2 & \text{if } 0 \leq x < 1 \\
  1 & \text{if } x \geq 1
\end{cases}
\]

Then \(\varphi \in \Phi\), but \(\varphi\) does not satisfy \(\varphi(t) = t \text{ if } t > 0\). We can find two sequences \(\{x_n\}\) and \(\{y_n\}\) such that \(x_n = 0\) and \(Tx_{n-1} = fx_n = y_n\).

When \(\gamma = 1\), we have \(y_n \to 1\), i.e. \(z = 1\) and \(Tz = T1 = 0\).

but \(d(z, Tz) \leq \varphi(d(z, Tz)) < d(z, Tz)\), a contradiction.

\(\Rightarrow \) \(z = Tz\), is not valid argument.

## 2 Main Results

We state our main definitions and theorem which may be regarded as a generalization to theorem 1.6.

**Definition 2.1:** Let \(\Phi\) denote the class of all functions \(\varphi : R^+ \to R^+\) satisfying:

for any \(\epsilon > 0\) there exists \(\delta > \epsilon\) such that \(\epsilon < t < \delta\) implie \(\varphi(t) \leq \epsilon\) and \(\varphi(t) = t \text{ if } t = 0\).

**Remark:** If \(\varphi \in \Phi\) and \(\varphi\) is continuous, then \(\varphi(t) \leq t \forall t > 0\).

The following example shows that if \(\varphi\) is not continuous, then the above remark may not hold.

**Example 2.2:** Define \(\varphi : R^+ \to R^+\) by \(\varphi(x) = \begin{cases} 
  \frac{1}{2} & \text{if } x < \frac{1}{2} \\
  \frac{1}{3} & \text{if } x > \frac{1}{2} \\
  1 & \text{if } x = \frac{1}{2}
\end{cases}\)

Then \(\varphi\) is discontinuous and \(\varphi(\frac{1}{2}) = 1 > \frac{1}{2}\).

**Examples:** (i) Define \(\varphi : R^+ \to R^+\) by \(\varphi(x) = Rx \forall x > 0\) and \(R\) is a positive real number less than 1.
(ii) Define $\varphi : R^+ \to R^+$ by $\varphi(x) = \begin{cases} \frac{2}{3} & \text{if } x < \frac{1}{2} \\ \frac{1}{2} & \text{if } x \geq \frac{1}{2} \end{cases}$

Then $\varphi$ is not continuous at $x = \frac{1}{2}$ and $\varphi(t) \leq \epsilon$.

(iii) Define $\varphi : R^+ \to R^+$ by $\varphi(x) = \begin{cases} \frac{2}{3} & \text{if } x < \frac{1}{2} \\ \frac{1}{2} & \text{if } x \geq \frac{1}{2} \end{cases}$

Then $\varphi$ is continuous at $x = \frac{1}{2}$ and $\varphi(t) \leq \epsilon$.

**Definition 2.3:** Let $(X, d)$ be a metric space and $T, f : X \to X$. Then they are said to be $\Phi$-weakly commuting if there exists a $\varphi \in \Phi$ such that $d(Tfx, fTx) \leq \varphi(d(Tx, fx)) \forall x \in X$.

**Definition 2.4:** Let $T$ and $f$ be maps on $X$ with values in a metric space $(X, d)$. Then $T$ is asymptotically regular with respect to $f$ at $x_0 \in X$ if there exists a sequence $\{x_n\}$ in $X$ such that $fx_n = Tx_{n-1}$, $n = 0, 1, 2, \ldots$ and $\lim_{n \to \infty} d(fx_n, fx_{n+1}) = 0$.

We state our main theorem, in which the conditions $(A_1)$ and $(A_3)$ of Theorem 1.6 are dropped.

**Theorem 2.5:** Let $T$ and $f$ be self maps on a complete metric space $(X, d)$ such that (2.5.1) $d(Tx, Ty) \leq \varphi(g(x,y))$ for all $x, y \in X$ where

$$g(x, y) = d(fx, fy) + \gamma[d(fx, Tx) + d(fy, Ty)], \gamma \geq 0 \text{ and } \varphi \in \Phi \text{ is continuous.}$$

(2.5.2) $T$ is asymptotically regular with respect to $f$ at $x_0 \in X$.

If $T$ is continuous then $T$ has a fixed point provided that $T$ and $f$ are $\Phi$-weakly commuting. Further if $f$ is continuous and $\gamma = 1$, then $T$ and $f$ have a unique common fixed point provided that $T$ and $f$ are $\Phi$-weakly commuting.

**Proof:** Since $T$ is asymptotically regular with respect to $f$ at $x_0$, there exists a sequence $\{x_n\}$ in $X$ such that $y_n = fx_n = Tx_{n-1}$, $n = 0, 1, 2, \ldots$ and $\lim_{n \to \infty} d(fx_n, fx_{n+1}) = \lim_{n \to \infty} d(y_n, y_{n+1}) = 0$.

Fix $\epsilon > 0$. There exists $\delta > \epsilon$ such that $\epsilon < t < \delta \Rightarrow \varphi(t) \leq t$.

Hence there exists $N \geq 1$ such that $d(y_n, y_{n+1}) < \frac{\delta - \epsilon}{1 + 2\gamma}$ for all $n \geq N$ .... (1)

We prove that $d(y_n, y_m) < \frac{\delta - \epsilon}{1 + 2\gamma} + \epsilon$ for all $m, n \geq N$, $m \geq n \geq N$ ....... (2)

Let $n \geq N$ be fixed.

If $m = n$ or $n + 1$, then clearly (2) holds, by (1).

Assume that (2) to hold for an integer $m \geq n + 1$

$$g(x_n, x_m) = d(fx_n, fx_m) + \gamma[d(fx_n, Tx_n) + d(fx_m, Tx_m)]$$

$$= d(y_n, y_m) + \gamma[d(y_n, y_{n+1}) + d(y_m, y_{m+1})]$$

$$< \frac{\delta - \epsilon}{1 + 2\gamma} + \epsilon + \gamma \left( \frac{\delta - \epsilon}{1 + 2\gamma} + \frac{\delta - \epsilon}{1 + 2\gamma} \right)$$

$$= \delta$$

If $d(Tx_n, Tx_m) \not< \epsilon$, then

$$\epsilon < d(Tx_n, Tx_m) < \varphi(g(x_n, x_m)) \leq g(x_n, x_m) < \delta$$

$$\Rightarrow \epsilon < g(x_n, x_m) \leq \epsilon,$$ a contradiction.
Therefore $d(Tx_n, Tx_m) \leq \epsilon$.
Consider $d(y_n, y_{n+1}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{m+1})$
\[ = d(y_n, y_{n+1}) + d(Tx_n, Tx_m) \]
\[ < \frac{\delta}{1+2\gamma} + \epsilon \]
Therefore $\{y_n\}$ is a Cauchy sequence.

Since $X$ is complete, $\{y_n\}$ is convergent, say, $z$.

(i) Suppose $T$ is continuous, then $TTx_n \to Tz$, $Tfx_n \to Tz$.

Since $T, f$ are $\Phi$-weakly commuting, we have

\[ d(Tfx_n, fTx_n) \leq \varphi(d(Tx_n, fx_n)) \leq d(Tx_n, fx_n) \]

On letting $n \to \infty$, we get $fTx_n \to Tz$.

If $z \neq Tz$, then by condition (2.5.1)

\[ d(Tx_n, TTx_n) \leq \varphi(g(x_n, Tx_n)) \]
\[ = \varphi(d(fx_n, fTx_n) + \gamma[d(fx_n, Tx_n) + d(TTx_n, TTx_n)]) \]

On letting $n \to \infty$, we get

\[ d(z, Tz) \leq \varphi(d(z, Tz)) < d(z, Tz), \text{ a contradiction.} \]

Therefore $Tz = z$.

(ii) If $f$ is continuous and $\gamma = 1$, then $ffx_n \to fz$ and $fTx_n \to fz$.

Since $T, f$ are $\Phi$-weakly commuting, we have

\[ d(Tfx_n, fTx_n) \leq \varphi(d(fx_n, Tx_n)) \leq d(fx_n, Tx_n) \]

Letting $n \to \infty$, we get $Tfx_n \to fz$.

If $fz \neq z$, by condition $(A_2)$, we have

\[ d(Tx_n, Tfx_n) \leq \varphi(g(x_n, fx_n)) \]

\[ = \varphi(d(fx_n, ffx_n) + \gamma[d(fx_n, Tx_n) + d(ffx_n, Tfx_n)]) \]

On letting $n \to \infty$

We get $d(z, fz) \leq \varphi(d(z, fz)) < d(z, fz)$, a contradiction.

Therefore $fz = z$.

Similarly we can prove that $Tz = z$.

Hence $fz = z = Tz$.

$\therefore$ $z$ is a common fixed point of $f$ and $T$.

To prove uniqueness, let $u$ be a common fixed point of $f$ and $T$.

Consider $d(z, u) = d(Tz, Tu) \leq \varphi(d(z, u))$
\[ = \varphi(d(fz, fu) + [d(fz, Tz) + d(fu, Tu)]) \]
\[ \Rightarrow d(z, u) \leq \varphi(d(z, u)) < d(z, u), \text{ a contradiction.} \]
Therefore $z = u$.
Hence $f$ and $T$ have unique common fixed point.

**Theorem 2.6:** Let $T$ and $f$ be self maps on an arbitrary non empty set $Y$ with values in a metric space $(X, d)$ such that

(2.6.1) $d(Tx, Ty) \leq \varphi(g(x, y))$ for all $x, y \in X$ where

$g(x, y) = d(fx, fy) + \gamma(d(fx, Tx) + d(fy, Ty))$, $0 \leq \gamma \leq 1$, $\varphi \in \Phi$

$
\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous.

(2.6.2) $T$ is asymptotically regular with respect to $f$ at $x_0 \in Y$.

If $f(Y)$ is a complete subspace of $X$, then

(i) $C(T, f)$ is non empty

Further, if $Y = X$, then

(ii) $T$ and $f$ have a unique common fixed point provided that $T$ and $f$ are $\Phi$-weakly commuting at a point $u \in C(T, f)$.

**Proof:** Since $T$ is asymptotically regular with respect to $f$ at $x_0$, there exists a sequence $\{x_n\}$ in $X$ such that

$y_n = fx_n = Tx_{n-1}$, $n = 0, 1, 2, \ldots$ and

$$\lim_{n \to \infty} d(fx_n, fx_{n+1}) = \lim_{n \to \infty} d(y_n, y_{n+1}) = 0.$$ 

Then $\{y_n\}$ is Cauchy sequence, which can be shown as in Theorem 2.5.

(i) Suppose $f(Y)$ is complete. Then the sequence $\{y_n\}$ is convergent. Say, $z$.

Let $u \in f^{-1}z$. Then $fu = z$.

By taking $x = u$ and $y = x_n$ in condition (2.6.1), we get

$$d(Tu, Tx_n) \leq \varphi(g(u, x_n))) = \varphi(d(fu, fx_n) + \gamma[d(fu, Tu) + d(fx_n, Tx_n)])$$

On letting $n \to \infty$, we get

$$d(Tu, Tz) \leq \varphi(d(z, z) + \gamma[d(z, Tu) + d(z, z)])$$

$$\Rightarrow d(Tu, Tz) \leq \varphi(\gamma d(z, Tz)) < d(z, Tz),$$ a contradiction

$$\therefore Tu = z = fu$$

Hence $C(T, f) \neq \emptyset$

(ii) Suppose $Y = X$ and $(T, f)$ is $\Phi$- weakly commuting at $u$.

Then $d(Tfu, fTu) \leq \varphi(d(fu, Tu)) = \varphi(0) = 0$.

So that $Tfu = fTu$ and $TTu = Tfu = f Tu = ffu$.

By taking $x = u$ and $y = Tu$ in condition (2.6.1), we get

$$d(Tu, TTu) \leq \varphi(g(u, Tu)) = \varphi(d(fu, fTu) + \gamma[d(fu, Tu) + d(fTu, TTu)])$$

$$\Rightarrow d(Tu, TTu) \leq \varphi(d(Tu, TTu)) < d(Tu, TTu),$$ a contradiction.
Therefore $TTu = Tu$ and $fTu = TTu = Tu = z$.
The uniqueness of common fixed point follows easily.

References


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