Approximate Amenability of Weighted Group Algebras

S. Naseri

Department of Mathematics, University of Kurdistan
Pasdaran boulevard, Sanandaj
Postal Code 66177-15175, P.O. Box 416, Iran
sabernaseri2008@gmail.com

Abstract

In this paper for a locally compact group $G$ we show that if $L^1(G, \omega)$ is approximately amenable then $G$ is an amenable group, but the converse is not valid in general. We also investigate the approximate amenability of $M(G, \omega)$ and $L^1(G, \omega)^{**}$.

Introduction

Recently Ghahramani and Loy in [3] introduced the notion of approximately amenable Banach algebras and, among other interesting results, they proved that for a locally compact group $G$, the group algebra $L^1(G)$ is approximately amenable if and only if $G$ is amenable.

The aim of the present paper is to show that for any weight function $\omega \geq 1$ on $G$, the approximate amenability of the Banach algebra $L^1(G, \omega)$ implies the amenability of $G$, but in general the converse is not true.

1 Preliminaries

Let $G$ denote a locally compact group with a fixed left Haar measurable $\lambda$; and $\omega$ be a weight function on $G$; that is a Borel measurable function $\omega: G \rightarrow \mathbb{R}^+$ such that

$$\omega(x,y) \leq \omega(x)\omega(y), \quad (x, y \in G).$$

The weighted group algebra $L^1(G, \omega)$ is the space of all measurable complex-valued functions $f$ on $G$ such that

$$\int_G |f(x)|\omega(x)dx < \infty,$$
and equipped with the convolution product $\ast$ of functions; that is for $f, g \in L^1(G, \omega)$ and $x \in G$

$$(f \ast g)(x) = \int_G f(xy^{-1})g(y)dy,$$

and the norm

$$\|f\|_\omega = \int_G |f(x)|\omega(x)dx.$$  

Also, let $L^\infty(G, \omega^{-1})$ be the space of all measurable complex-valued functions $\phi$ on $G$, such that $\frac{\phi}{\omega}$ is essentially bounded, and for $\phi \in L^\infty(G, \omega^{-1})$ define

$$\|\phi\|_{\infty, \omega} = \|\frac{\phi}{\omega}\|_{\infty} = \text{ess sup}\{\frac{|\phi(x)|}{\omega(x)}| x \in G\}.$$  

The spaces $L^1(G, \omega)$ and $L^\infty(G, \omega^{-1})$ are in duality by

$$\langle f, \phi \rangle = \int_G f \phi d\lambda \quad (f \in L^1(G, \omega), \phi \in L^\infty(G, \omega^{-1})),$$

and if $\omega(x) \geq 1, (x \in G)$ then $L^\infty(G) \subseteq L^\infty(G, \omega^{-1})$. Let $M(G, \omega)$ be the Banach space of all complex-valued, regular Borel measures $\mu$ on $G$ such that

$$\|\mu\|_\omega = \int_G \omega(x)d|\mu|(x) < \infty.$$  

Note that $M(G, 1) = M(G)$. If $\omega(x) \geq 1(x \in G)$ then $M(G, \omega)$ is a subalgebra of $M(G)$.

Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \to X$ is called a derivation if

$$D(ab) = D(a).b + a.D(b) \quad (a, b \in \mathcal{A}).$$

For every $x \in X$ we define $ad^A_x$ by

$$ad^A_x(a) = ax - xa \quad (a \in \mathcal{A}).$$

It is easily seen that $ad^A_x$ is a derivation. Derivation of this form are called inner derivation.

Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. Then $X^*$, the dual space of $X$, is a Banach $\mathcal{A}$-bimodule for operations given by

$$\langle x, a\xi \rangle = \langle xa, \xi \rangle, \quad \langle x, \xi a \rangle = \langle ax, \xi \rangle \quad (a \in \mathcal{A}, \ x \in X, \ \xi \in X^*),$$

$X^*$ is the dual module of $X$; and in particular $\mathcal{A}^*$ is the dual module of $\mathcal{A}$.
A Banach algebra $\mathcal{A}$ is called *amenable* if for any $\mathcal{A}$-bimodule $X$, any derivation $D : \mathcal{A} \to X^*$ is inner. This definition of amenability was introduced by Johnson in [5]. A Banach algebra $\mathcal{A}$ is called *weakly amenable* if any derivation $D : \mathcal{A} \to A^*$ inner. Trivially, any amenable Banach algebra is weakly amenable.

Let $G$ be a locally compact group, a *mean* on $L^\infty(G)$ is a positive functional $m \in L^\infty(G)^*$ such that 
$$\langle 1, m \rangle = \|m\| = 1.$$ 
A mean $m$ on $L^\infty(G)$ is called *left invariant* if
$$\langle \delta_x \ast \phi, m \rangle = \langle \phi, m \rangle \quad (x \in G, \ \phi \in L^\infty(G)).$$
A locally compact group $G$ is called *amenable* if there exists a left invariant mean on $L^\infty(G)$. Note that $G$ is amenable if and only if $L^1(G)$ is an amenable Banach algebra.

A derivation $D : \mathcal{A} \to X$ is called *approximately inner* if there exists net $(\xi_\alpha) \subseteq X$ such that for every $a \in A$,
$$D(a) = \text{norm} \lim_{\alpha} (a \cdot \xi_\alpha - \xi_\alpha \cdot a).$$
Recall from [3] that a Banach algebra $\mathcal{A}$ is called *approximately amenable* if for any $\mathcal{A}$-bimodule $X$, any derivation $D : \mathcal{A} \to X^*$ is approximately inner.

## 2 Approximate amenability of weighted group algebras

Recently, Ghahramani and Loy in [3] proved that for a locally compact group $G$ the Banach algebra $L^1(G)$ is approximately amenable if and only if $G$ is amenable. In this section we prove that if $\omega \geq 1$ is any weight function on $G$ and $L^1(G, \omega)$ is approximately amenable, then $G$ is amenable. Through an example we show that the converse is not valid in general.

**Theorem 2.1.** Let $G$ be a locally compact group, and let $\omega$ be a weight function on $G$ such that $\omega(x) \geq 1 \ (x \in G)$. If $L^1(G, \omega)$ is approximately amenable, then $G$ is amenable.

**Proof.** By the following operations, $L^\infty(G, \omega^{-1})$ is a Banach $M(G, \omega)$-bimodule;
$$\langle f, \phi, \mu \rangle = \langle \mu \ast f, \phi \rangle, \quad \mu \cdot \phi = \mu(G)\phi,$$
where $\mu \in M(G, \omega)$, $\phi \in L^\infty(G, \omega^{-1})$ and $f \in L^1(G, \omega)$. Note that we have
$$\langle \mu \ast f, \phi \rangle = \int_G f(y^{-1}x)d\mu(y).$$
for $\mu \in M(G, \omega)$, $f \in L^1(G, \omega)$ and $x \in G$. Since for every $\mu \in M(G, \omega)$ and $f \in L^\infty(G, \omega^{-1})$

$$\langle f, 1, \mu \rangle = \langle \mu \ast f, 1 \rangle = \int (\mu \ast f)(x)dx$$

$$= \int \int f(y^{-1}x)d\mu(y)dx$$

$$= \int \int f(x)d\mu(y)$$

$$= \mu(G) \int f(x)dx$$

$$= \mu(G) \langle f, 1 \rangle.$$  

Thus $1, \mu = \mu(G).1 \in C1$; and by definition we have $\mu.1 \in C1$, that is the space $C1$ is a submodule of $M(G, \omega)$. If we set $Z = L^\infty(G, \omega^{-1})/C1$ then $Z$ is a $M(G, \omega)$-bimodule, with $Z^* = \{ m \in L^\infty(G, \omega^{-1})^* | m(1) = 0 \}$. By Hahn Banach Theorem there is $\nu \in L^\infty(G, \omega^{-1})^*$ such that $\nu(1) = 1$. It is easy to prove that the mapping $\tilde{D} : M(G, \omega) \rightarrow Z^*$, defined by

$$\tilde{D}(\mu)(\phi + C1) = (\mu.\nu - \mu(G)\nu)(\phi) \quad (\mu \in M(G, \omega), \phi \in L^\infty(G, \omega^{-1})),$$

is a well-defined derivation. If the restriction $\tilde{D}$ to $L^1(G, \omega)$ is denoted by $D$, then form the approximate amenability of $L^1(G, \omega)$ it follows that there is a net $(m_i)$ in $Z^*$ such that for every $\mu \in L^1(G, \omega)$ we have

$$D(\mu) = \mu.\nu - \nu\mu(G) = \lim_i (\mu.m_i - m_i\mu(G)).$$

For $\mu \in L^1(G, \omega)$ with $\mu(G) = 1$ and $\phi \in L^\infty(G, \omega^{-1})$, $\mu.\phi = \phi$. Hence for every $x \in G$, $\phi \in L^\infty(G, \omega^{-1})$ we have

$$\langle \phi + C1, D(\delta_x).\mu \rangle = \langle \mu(\phi + C1), D(\delta_x) \rangle$$

$$= \langle \mu.\phi + C1, D(\delta_x) \rangle$$

$$= \langle \phi + C1, D(\delta_x) \rangle.$$  

Thus $D(\delta_x).\mu = D(\delta_x)$. Therefore for every $x \in G$ and $\mu \in L^1(G, \omega)$ with $\mu(G) = 1$ we have

$$\delta_x.\nu - \nu = D(\delta_x) = D(\delta_x).\mu$$

$$= D(\delta_x \ast \mu) - \delta_x D(\mu)$$

$$= \lim_i [(\delta_x \ast \mu).m_i - m_i(\delta_x \ast \mu) - \delta_x(\mu.m_i - m_i.\mu)]$$

$$= \lim_i [(\delta_x \ast \mu).m_i - m_i - \delta_x(\mu.m_i - m_i)]$$

$$= \lim_i (\delta_x m_i - m_i).$$
Thus
\[ \lim_i \delta_x(\nu - m_i) - (\nu - m_i) = 0. \quad (\star) \]
Since \( \langle \nu - m_i, 1 \rangle = 1 \), it follows that \( \lim_i \| \nu - m_i \| \neq 0 \), so there exists a subnet \((m_j)\) of \((m_i)\) such that \( \| \nu - \tilde{m}_j \| \neq 0 \). For every \( j \) if we set
\[ \tilde{n}_j = \frac{\nu - \tilde{m}_j}{\| \nu - \tilde{m}_j \|}, \]
then \( \tilde{n}_j \in L^\infty(G, \omega^{-1}) \). If for each \( j \) the restriction of \( \tilde{n}_j \) to \( L^\infty(G) \) is denoted by \( n_j \), then \( \| n_j \| = 1 \) and by \((\star)\) we have norm-lim \( \| \delta_x.n_j - n_j \| = 0 \), for every \( x \in G \). Since the space \( L^\infty(G) \) is a commutative \( C^* \)-algebra with identity \( 1 \), so there is a compact space \( T \) such that \( L^\infty(G)^* = M(T) \). Since for each \( x \in G \) and \( m \in M(T) \), we have \( |\delta_x.m| = \delta_x|m| \), therefore
\[ |\delta_x.n_j - |n_j|| = |\delta_x.n_j - |n_j|| \leq |\delta_x.n_j - n_j| \quad (x \in G), \]
and since \( \lim_j |\delta_x.n_j - n_j| = 0 \), thus for each \( x \in G \)
\[ \text{norm - lim}_j (\delta_x.n_j - |n_j|) = 0. \quad (\star\star) \]
Let \( n \) be a weak\(^*\) - cluster point of \((n_j)\). Clearly \( n \) is positive, and by \((\star\star)\) for each \( x \in G \) we have
\[ \delta_x.n = n, \]
and
\[ n(1) = \lim_j |n_j|(1) = |n_j|(G) = \| n_j \| = 1. \]
Therefore \( n \) is a left invariant mean on \( L^\infty(G) \), and thus \( G \) is an amenable group.

In the following lemma we show that an approximately amenable commutative Banach algebra is weakly amenable.

**Lemma 2.2.** Let \( A \) be a commutative Banach algebra. If \( A \) is approximately amenable, then \( A \) is weakly amenable.

**Proof.** Suppose \( D : A \to A^* \) is a derivation; we show that \( D \) is inner. Since \( A \) is approximately amenable and commutative, so there exists a net \((\xi_i)\) in \( A^* \) such that for each \( a \in A \)
\[ D(a) = \text{norm - lim}_i (a.\xi_i - \xi_i.a) = 0. \]
On the other hand since \( A \) is commutative, every inner derivation on \( A \) is zero; so \( D \) is inner. Therefore \( A \) is weakly amenable. \( \square \)
The Example 6.2 of [3] shows that an approximately amenable Banach algebra need not be weakly amenable.

The following example shows that the converse of Theorem 2.1 is not true in general.

**Example 2.3.** If we define the weight function $\omega$ on $(\mathbb{Z}, +)$ by $\omega(n) = 1 + |n|$ for every $n \in \mathbb{Z}$, then the Banach algebra $l^1(\mathbb{Z}, \omega)$ is not approximately amenable.

**Proof.** Since $(\mathbb{Z}, +)$ is a commutative group, so is amenable. Let

$$l^1(\mathbb{Z}, \omega) = \{a = (a(n) : n \in \mathbb{Z}) | \sum_{-\infty}^{\infty} |a(n)|\omega(n) \leq \infty\}.$$  

Then $l^1(\mathbb{Z}, \omega)$ is a commutative Banach algebra with respect to convolution multiplication

$$(a * b)(n) = \sum_{k=-\infty}^{\infty} a(n-k)b(k) \quad (n \in \mathbb{Z}, a, b \in l^1(\mathbb{Z}, \omega)), $$

and the norm

$$\|a\| = \sum_{-\infty}^{\infty} |a(n)|\omega(n) < \infty \quad (a \in l^1(\mathbb{Z}, \omega)).$$

We show that $l^1(\mathbb{Z}, \omega)$ is not approximately amenable. To see this note that

$$\sup\{\frac{\omega(n+m)}{\omega(n)\omega(m)} \cdot \frac{1 + |n|}{1 + |n + m|} | n, m \in \mathbb{Z}\} = \sup\{\frac{1}{1 + |m|} | m \in \mathbb{Z}\} = 1.$$  

So by Theorem 2.3 of [1], $l^1(\mathbb{Z}, \omega)$ is not weakly amenable. On the other hand since $l^1(\mathbb{Z}, \omega)$ is a commutative Banach algebra; and if it is approximately amenable, so by Lemma 2.2 is weakly amenable; and this is a contradiction. So $l^1(\mathbb{Z}, \omega)$ is not approximately amenable.

Let $\omega$ be a weight function on $G$, for $x \in G$ we the define $\omega^*(x) = \omega(x)\omega(x^{-1})$. In the following theorem we prove that if $\omega^*$ is bounded on $G$ then the converse of Theorem 2.1 is true.

**Theorem 2.4.** Suppose that $\omega$ a weight function on a locally compact group $G$ such that $\omega \geq 1$ and $\omega^*$ is bounded. Then $G$ is amenable if and only if $L^1(G, \omega)$ is approximately amenable.

**Proof.** Since $G$ is amenable, and $\omega^*$ is bounded on $G$, by Theorem 0 of [4] $L^1(G, \omega)$ is amenable, so $L^1(G, \omega)$ is approximately amenable. \qed
3 Approximate amenability \( M(G, \omega) \) and \( L^1(G, \omega)^{**} \) of a locally compact group \( G \)

It is standard that \( L^1(G) \) always has a bounded approximate identity which is a net consisting of continuous functions of compact support, and this net is clearly also a bounded approximate identity for \( L^1(G, \omega) \).

Let \( \omega \) be a weight function on \( G \) with \( \omega(x) \geq 1 \) (\( x \in G \)), then with the convolution product \(*\) given by

\[
\langle \phi, \mu * \nu \rangle = \int_G \int_G \phi(st) d\mu(s) d\nu(t) \quad (\mu, \nu \in M(G, \omega), \ \phi \in C_0(G, \omega)),
\]

the Banach space \( M(G, \omega) \) defines a unital convolution Banach algebra for which \( L^1(G, \omega) \) is a closed ideal.

In the following lemma we show that if \( M(G, \omega) \) is approximately amenable then \( G \) is amenable.

**Theorem 3.1.** Let \( G \) be a locally compact group. If \( M(G, \omega) \) is approximately amenable, then \( G \) is amenable.

**Proof.** Since \( L^1(G, \omega) \) is a closed two ideal of \( M(G, \omega) \) and has an approximate identity, from Corollary 2.3 of [3] it follows that \( L^1(G, \omega) \) is approximately amenable. So by Theorem 2.1, \( G \) is amenable. \(\square\)

It is well known (c.f. [6]) that if \( G \) is amenable and \( \omega^* \) is bounded on \( G \), then there is a continuous positive character \( \phi \) on \( G \) such that

\[
\phi \leq \omega \leq c\phi \quad \text{that} \quad c = \sup\{\omega^*(x) | x \in G\}.
\]

In particular

\[
L^1(G, \omega) = L^1(G, \phi) \simeq L^1(G), \quad \text{and} \quad M(G, \omega) = M(G, \phi) \simeq M(G).
\]

**Theorem 3.2.** Let \( G \) be a discrete amenable group and \( \omega^* \) be bounded, then \( M(G, \omega) \) is approximately amenable.

**Proof.** Since \( G \) is amenable and \( \omega^* \) is bounded, it follows that \( M(G, \omega) \simeq M(G) \). Since \( G \) is discrete, from Theorem 1.1 of [2], it follows that \( M(G) \) is amenable and therefore is approximately amenable. So \( M(G, \omega) \) is approximately amenable. \(\square\)

**Proposition 3.3.** If \( M(G, \omega) \) is approximately amenable and \( \omega^* \) bounded on \( G \), then \( G \) is a discrete group.

**Proof.** Since \( M(G, \omega) \) is approximately amenable, from Theorem 3.1 we conclude that \( G \) is amenable. Using the fact that \( \omega^* \) is bounded, we infer that \( M(G, \omega) \simeq M(G) \). So by Theorem 1.1 of [2], \( G \) is discrete. \(\square\)
References


Received: June, 2010