In this note the authors show that Kepler’s quartic curve with eccentricity $e$ has $\lambda$-invariant $e^2$ as an elliptic curve.

 mathematics Subject Classification: 14H45, 37N05, 85-03

Key words elliptic curve, planetary orbit

1. Introduction

In the paper [8], a birational transformation was given for Kepler’s quartic curve. By such a transformation, Kepler’s quartic curve was transformed into a cubic elliptic curve. In this paper we study invariants of the Kepler’s quartic curve as an elliptic curve. Many authors mentioned that Johannes Kepler supposed an ”ovoid” as Mars’s orbit around the sun before he found that an ellipse was the true orbit of Mars (cf. [2], [3], [4], [5], [11], [12]). The quartic
curve is the most famous one. The first rigid formulation of Kepler’s quartic curve was provided by Fladt in [5] (cf. [11]).

On the Euclidean plane, we shall use Cartesian coordinates \((X, Y)\). We use the notations in [4] except for \(K\). The points \(C = (a + ae, 0), D = (-a + ae, 0)\) are respective the aphelion and the perihelion of a planet, where \(0 < a\) is the radius of the deferent and \(ae\) is the radius of the epicycle with \(0 < e < 1\). The parameter \(\epsilon = 2e/(1 + e^2)\) also plays a role similar to the eccentricity of the orbit. However Fladt adopted \(\epsilon\) as the eccentricity of the planet in [5], we call \(e = (1 - \sqrt{1 - \epsilon^2})/\epsilon\) the eccentricity of the planet. The Sun is fixed at the origin \(A = (0, 0)\). The planet lies on a circle

\[
(X - a \cos \beta)^2 + (Y - a \sin \beta)^2 = a^2e^2, \tag{1.1}
\]

where the parameter \(\beta\) is said to be the eccentric anomaly, this circle is called the epicycle. However the position of the planet is given by \((a \cos \beta + ae, a \sin \beta)\) in the Ptolemaic model, Kepler supposed a new model. Denote by \(Z\) the center \((a \cos \beta, a \sin \beta)\) of the above circle. Denote by \(B\) the center \((ae, 0)\) of the eccentric circle \((X - ae)^2 + Y^2 = a^2e^2\). Suppose that \(K\) is the intersection of the eccentric circle \((X - ae)^2 + Y^2 = a^2e^2\) and the straight line \(AZ\), where we assume that \(\overrightarrow{AK} = \lambda \overrightarrow{AZ}\) with \(\lambda > 0\). The point \(V = (ax, ay)\) of the planet corresponding to \(\beta\) lies on the circle (1.1) and the lengths of \(AK\) and \(AV\) are equal. We assume \(0 < |\beta| < \pi\). There are two points satisfying the conditions for \(V\). We choose \((x, y)\) for which the modulus of the angle \(\angle CAV\) is less than that of another. By some computations, we find that the point \(V = (ax, ay)\) is given by

\[
x = \cos \beta + e \sqrt{1 - e^2 \sin^2 \beta}, \quad y = (1 - e^2) \sin \beta. \tag{1.2}
\]

We assume that \(0 < \beta < \pi\). Under this condition the three angles \(\angle CAV, \angle CAK, \angle CBK\) satisfy the equation

\[
\angle CBK - \angle CAK = \angle CAK - \angle CAV (> 0). \tag{1.3}
\]

The parameter \(\beta\) represents \(\angle CAK\). This equation (1.3) was mentioned by Kepler himself in [6], Ch. 30 (cf. [7],[8]). The geometric construction of the orbit based on the equation (1.2) was given in [4], Section 6. We present a graphic of the upper part \((0 \leq \beta \leq \pi)\) of the orbit for \(a = 1, e = 0.6\) in Figure 1.

As it was mentioned in [8], the point \((x, y)\) with parameter representation (1.2) satisfies an algebraic equation \(F(1, x, y) = 0\) for the form

\[
F(t, x, y; e) = (1 - e^2)^2x^4 + 2(1 + e^4)x^2y^2 + (1 + e^2)^2y^4 - 2(1 - e^2)^2(1 + e^2)t^2(x^2 + y^2) + (1 - e^2)^4t^4 = 0. \tag{1.4}
\]
Invariants of Kepler’s curve

We study the properties of the curve $F(t, x, y; e) = 0$ which are invariant under birational transformations. We define a complex projective plane curve $C_e$ in the complex projective plane $\mathbb{CP}^2$ by

$$C_e = \{(t, x, y) \in \mathbb{CP}^2 : F(t, x, y; e) = 0\}, \quad (1.5)$$

for $0 < e < 1$. We present a graphic of the real affine part of $C_e$ for $e = 0.6$ in Figure 2.

2. Computation of the invariants of the elliptic curve

The curve $C_e$ has two ordinary double points at

$$(t, x, y) = (1, 0, \frac{1-e^2}{\sqrt{1+e^2}}), \quad (t, x, y) = (1, 0, -\frac{1-e^2}{\sqrt{1+e^2}}).$$
The curve $C_e$ has no other singular point.

An irreducible complex projective plane curve $F(t, x, y) = 0$ is called an elliptic curve if its genus is 1. This is equivalent to the following condition: The curve $F(t, x, y) = 0$ is transformed into a non-singular cubic curve

$$G(T, X, Y) = -TY^2 + a_0(X - \alpha T)(X - \beta T)(X - \gamma T) = 0$$

with $a_0(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) \neq 0$ by a birational transformation. It is shown in [8] that the form $F(t, x, y : e)$ is irreducible in the polynomial ring $\mathbb{C}[t, x, y]$.

Following [10] pp. 489-493, we express the curve $C_e$ as the image of a cubic elliptic curve under a birational transformation. We define a correspondence $(t, x, y) \mapsto (t_1, x_1, y_1)$ of $\mathbb{CP}^2$ onto itself by the relations

$$t_1 = tx, \quad x_1 = -(1 + e^2)(x + iy)x, y_1 = (1 - e^2)^2t^2 + i(1 + e^2)xy - (1 + e^2)y^2.$$

For finite many points $(t, x, y)$ of $\mathbb{CP}^2$ is associated with the point $(0, 0, 0)$ by the above rule. For instance, the point $(t, x, y) = (0, 1, i)$ is associated with $(0, 0, 0)$. So the above correspondence is not exactly a map of $\mathbb{CP}^2$ onto itself. We adopt some exceptional points. Its inverse correspondence is given by the following

$$t = (1 + e^2)t_1(x_1 - y_1), x = -(1 - e^2)^2(1 + e^2)t_1^2 - x_1^2,$$
$$y = -i \{(1 - e^2)^2(1 + e^2)t_1^2 + x_1y_1\}.$$

Substituting these equations into $F(t, x, y; e) = 0$, we have the equation

$$L(t_1, x_1, y_1; e) = 8e^2(1 - e^2)^2(1 + e^2)t_1^2x_1$$
$$+(1 + e^2)^2y_1^3 + (1 + e^2)^2x_1y_1^2 - (1 - e^2)^2x_1^2y_1 - (1 - e^2)^2x_1^3 = 0, \quad (2.1)$$

(cf. [10]). To transform it into Weierstrass’s canonical form, we change the coordinates as the following:

$$t_1 = \frac{i\sqrt{1 + e^2}}{4\sqrt{2e(1 - e^2)}}Y, \quad y_1 = X - \frac{x_1}{3}, \quad x_1 = x_1.$$

Then the curve $L(t_1, x_1, y_1 : e) = 0$ is transformed into the curve

$$x_1Y^2 = 4X^3 - g_2x_1^2X - g_3x_1^3 = 4(X - e_1x_1)(X - e_2x_1)(X - e_3x_1), \quad (2.2)$$

where

$$e_1 = \frac{2(2 - e^2)}{3(1 + e^2)}, \quad e_2 = \frac{2(2e^2 - 1)}{3(1 + e^2)}, \quad e_3 = \frac{-2}{3}, \quad (2.3)$$
and
\[ g_2 = \frac{16(1 - e^2 + e^4)}{3(1 + e^2)^2}, \quad g_3 = \frac{-32(2 - e^2)(2e^2 - 1)}{27(1 + e^2)^2}. \] (2.4)

In the above, the roots \( e_1, e_2, e_3 \) of the equation \( 4X^3 - g_2X - g_3 = 0 \) are real numbers and satisfy \( e_3 < e_2 < e_1 \). The \( \lambda \)-invariant of the curve (2.2) is defined by
\[ \lambda = \frac{e_1 - e_2}{e_1 - e_3}. \] (2.5)

This invariant is also denoted by \( \chi^2 \) (cf. [1], [9]). It follows from (2.3), (2.5) that
\[ 0 < \lambda = e^2 < 1. \]

If we consider another coordinate system \( (\tilde{Y}, \tilde{X}, x = 1) \) defined by
\[ \tilde{X} = -X, \tilde{Y} = iY, \]
then the quantity \( \lambda = e^2 \) is replaced by \( 0 < \tilde{\lambda} = 1 - \lambda = 1 - e^2 < 1 \). The following invariant \( J \) is also often discussed.
\[ J = \frac{g_3^3}{g_2^3 - 27g_3^2}, \] (2.6)
(cf. [1], [9]). It follows from (2.4), (2.6) that
\[ J = \frac{4(1 - e^2 + e^4)^3}{27e^3(1 - e^2)^2}. \] (2.7)

The affine curve \( Y^2 = 4X^3 - g_2X - g_3 \) in the complex affine space \( \mathbb{C}^2 \) is parametrized as
\[ X = \mathcal{P}(s : g_2, g_3), \quad Y = \mathcal{P}'(s : g_2, g_3), \]
by the Weierstass \( P \)-function \( \mathcal{P} \) corresponding to the coefficients \( g_2, g_3 \) and its derivative function. In the case \( e_1 > e_2 > e_3 \), the group of the periods of the elliptic function \( \mathcal{P} \) is generated by a real number \( 2\omega_1 \) and a pure imaginary number \( 2\omega_3 \) given by
\[ \omega_1 = \int_{e_1}^{+\infty} \frac{1}{\sqrt{4(x - e_1)(x - e_2)(x - e_3)}} dx = \int_{e_3}^{e_2} \frac{1}{\sqrt{4(x - e_1)(x - e_2)(x - e_3)}} dx, \]
\[ \omega_3 = i \int_{-\infty}^{e_3} \frac{1}{\sqrt{-4(x - e_1)(x - e_2)(x - e_3)}} dx = i \int_{e_2}^{e_1} \frac{1}{\sqrt{-4(x - e_1)(x - e_2)(x - e_3)}} dx. \]
The $\tau$-invariant of the curve (2.2) is defined as the ratio $\omega_3/\omega_1$ of these 2 half-periods. It is a purely imaginary number in this case. This invariant is also computed by using the formula

$$\mathcal{P}(s : g_2, g_3) = e_1 + (e_1 - e_3) \frac{Cn^2(w : \lambda)}{Sn^2(w : \lambda)}$$

for the Jacobi $Sn$ and $Cn$, where $w = \sqrt{e_1 - e_3 s}$. It follows that the ratio $\tau = \omega_3/\omega_1$ is obtained as the ratio

$$\tau = \frac{iK(1 - \lambda)}{K(\lambda)}$$

for the complete elliptic integrals

$$K(\lambda) = \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-\lambda x^2)}} dx.$$ 

References


Received: Month xx, 2010