On Right Fixed Maps of $Q$-Algebras

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Abstract
In this paper, the concept of a right fixed map in an $Q$-algebra is discussed and some fundamental properties to $Q$-algebras are discussed.

Mathematics Subject Classification: 06F35, 03G25, 08A30

Keywords: $Q$-algebra, self-distributive, filter, right fixed map, commutative, idempotent

1 Introduction
Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Q. P. Hu and X. Li introduced a wide class of abstracts: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [5], J. Neggers, etc introduced the notion of an $Q$-algebra as a dualization of a generation of a BCK/BCI-algebras. In this paper, the concept of a right fixed map in an $Q$-algebra is discussed and some fundamental properties to $Q$-algebras are discussed.

2 Preliminaries
By an $Q$-algebra we mean an algebra $(X; *, 0)$ of type $(2, 0)$ with a single binary operation “*” that satisfies the following identities: for any $x, y, z \in X$, $x * (y * z) = (x * y) * z$ and $x * 0 = x = 0 * x$.
(Q1) \( x \ast x = 0 \) for all \( x \in X \),
(Q2) \( x \ast 0 = x \) for all \( x \in X \),
(Q3) \( (x \ast y) \ast z = (x \ast z) \ast y \) for all \( x, y, z \in X \).

We introduce a relation “\( \leq \)’” on \( X \) by \( x \leq y \) imply \( x \ast y = 0 \). An Q-algebra \((X, \ast, 0)\) is said to be self-distributive if \((x \ast y) \ast z = (x \ast z) \ast (y \ast z)\) for all \( x, y, z \in X \). A non-empty subset \( S \) of an Q-algebra \( X \) is said to be a subalgebra of \( X \) if \( x \ast y \in S \) whenever \( x, y \in S \).

In an Q-algebra, the following identities are true:

(p1) \( x \ast ((x \ast y) \ast y) = 0 \).

**Definition 2.1.** Let \((X, \ast, 0)\) be an Q-algebra and \( I \) a non-empty subset of \( X \). Then \( I \) is said to be an ideal of \( X \) if

(F1) \( 0 \in I \),
(F2) If \( x \ast y \in I \) and \( y \in I \) imply \( x \in I \).

**Example 2.2.** Let \( X = \{0, a, b, c\} \) in which “\( \ast \)” is defined by

\[
\begin{array}{c|cccc}
  \ast & 0 & a & b & c \\
  \hline
  0 & 0 & 0 & 0 & 0 \\
  a & a & 0 & 0 & 0 \\
  b & b & 0 & 0 & 0 \\
  c & c & c & c & 0 \\
\end{array}
\]

It is easy to know that \( X \) is an Q-algebra, and \( I = \{0, a, b\} \) is an ideal of \( X \).

We consider the following property (S) where (S) means

(S) \( 0 \ast x = 0 \).

**3 Right fixed maps in Q-Algebras**

In what follows, let \( X \) denote an Q-algebra with (S) unless otherwise specified.

**Definition 3.1.** A right fixed map \( \alpha \) of \( X \) is defined to be a self map \( \alpha : X \to X \) satisfying \( \alpha(x \ast y) = \alpha(x) \ast y \) for all \( x, y \in X \).

**Example 3.2.** Let \( X = \{0, a, b\} \) in which “\( \ast \)” is defined by
On right fixed maps of Q-algebras

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Then $X$ is an Q-algebra. It can be easily very that the self map $\alpha$ of $X$ defined by $\alpha(0) = 0, \alpha(a) = 0$ and $\alpha(b) = b$ is a right fixed map.

**Example 3.3.** Let $X = \{0, a, b, c\}$ in which “$\ast$” is defined by

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Let $\alpha : X \to X$ be defined by $\alpha(0) = 0, \alpha(a) = 0, \alpha(b) = c$ and $\alpha(c) = c$. Then $\alpha$ is not a right fixed map of $X$ since $\alpha(b \ast a) \neq \alpha(b) \ast a$.

**Lemma 3.4.** If $\alpha$ is a right fixed map of $X$, then we have

1. $\alpha(0) = 0$,
2. $\alpha(x) \leq x$ for all $x \in X$,
3. $x \leq y$ implies $\alpha(x) \leq y$ for all $x, y \in X$.

**Proof.** (1) For every $x, y \in X$, we have

$$\alpha(0) = \alpha(0 \ast \alpha(0)) = \alpha(0) \ast \alpha(0) = 0.$$  

(2) For every $x \in X$, we have $0 = \alpha(0) = \alpha(x \ast x) = \alpha(x) \ast x$, and so $\alpha(x) \leq x$.

(3) Suppose that $x \leq y$ for every $x, y \in X$. Then

$$0 = \alpha(0) = \alpha(x \ast y) = \alpha(x) \ast y,$$

and so $\alpha(x) \leq y$.

**Proposition 3.5.** If $\alpha$ and $\beta$ are right fixed maps of $X$, then $\alpha \circ \beta$ is a right fixed map of $X$.

**Proof.** Let $\alpha$ and $\beta$ be right fixed maps of $X$. Then

$$(\alpha \circ \beta)(x \ast y) = \alpha(\beta(x \ast y)) = \alpha(\beta(x) \ast y) = \alpha(\beta(x)) \ast y = (\alpha \circ \beta)(x) \ast y$$

for all $x, y \in X$.
**Theorem 3.6.** Let $\alpha$ be a right fixed map of $X$. Then $\alpha$ is one-to-one if and only if $\alpha$ is the identity map.

*Proof.* Sufficiency is obvious. Suppose that $\alpha$ is one-to-one. For $x \in X$, we have

$$\alpha(x \ast \alpha(x)) = \alpha(x) \ast \alpha(x) = 0 = \alpha(0)$$

and so $x \ast \alpha(x) = 0$, i.e., $x \leq \alpha(x)$. Since $\alpha(x) \leq x$ for all $x \in X$, it follows that $\alpha(x) = x$ so that $\alpha$ is the identity map. \qed

Let $\alpha$ be a right fixed map of $X$. Define a set $F$ by

$$F := \{ x \mid \alpha(x) = x \}$$

for all $x \in X$.

**Proposition 3.7.** Let $\alpha$ be a right fixed map of $X$. Then $F$ is a subalgebra of $X$.

*Proof.* Clearly, $0 \in F$ and so $F$ is nonempty. Let $x, y \in F$. Then we have $\alpha(x) = x$ and $\alpha(y) = y$, and so

$$\alpha(x \ast y) = \alpha(x) \ast y = x \ast y.$$ 

This implies $x \ast y \in F$. Hence $F$ is a subalgebra of $X$. \qed

**Proposition 3.8.** Let $\alpha$ be a right fixed map of $X$. If $x \in F$, then we have $(\alpha \circ \alpha)(x) = x$.

*Proof.* Let $x \in F$. Then we have

$$(\alpha \circ \alpha)(x) = \alpha(\alpha(x)) = \alpha(x) = x.$$ 

This completes the proof. \qed

Let $\alpha$ be a right fixed map of $X$. Define a $\text{Ker}(\alpha)$ by

$$\text{Ker}(\alpha) = \{ x \mid \alpha(x) = 0 \}$$

for all $x \in X$.

**Proposition 3.9.** Let $\alpha$ be a right fixed map of $X$. Then $\text{Ker}(\alpha)$ is a subalgebra of $X$. 
Proof. Clearly, $0 \in \text{Ker}(\alpha)$, and so $\text{Ker}(\alpha)$ is nonempty. Let $x, y \in \text{Ker}(\alpha)$. Then $\alpha(x) = 0$ and $\alpha(y) = 0$. Hence we have
\[ \alpha(x \ast y) = \alpha(x) \ast y = 0 \ast y = 0, \]
and so $x \ast y \in \text{Ker}(\alpha)$. Thus $\text{Ker}(\alpha)$ is a subalgebra of $X$.

An $Q$-algebra $X$ is said to be commutative if for all $x, y \in X$,
\[ x \ast (x \ast y) = y \ast (y \ast x). \]

Proposition 3.10. Let $X$ be a commutative $Q$-algebra. If $x \in \text{Kerd}$ and $x \leq y$, then we have $y \in \text{Ker}(\alpha)$.

Proof. Let $x \in \text{Ker}(\alpha)$ and $y \leq x$. Then $\alpha(x) = 0$ and $x \ast y = 0$.
\[
\begin{align*}
\alpha(y) &= \alpha(y \ast 0) = \alpha(y \ast (y \ast x)) \\
&= \alpha((x \ast (x \ast y)) \\
&= \alpha(x) \ast (x \ast y) \\
&= 0 \ast (x \ast y) \\
&= 0,
\end{align*}
\]
and so $y \in \text{Ker}(\alpha)$. This completes the proof.

Proposition 3.11. Let $\alpha$ be a right fixed map of $X$ and an endomorphism. Then $\text{Ker}(\alpha)$ is an ideal of $X$.

Proof. Clearly, $0 \in \text{Ker}(\alpha)$. Let $y \in \text{Ker}(\alpha)$ and $x \ast y \in \text{Ker}(\alpha)$. Then we have $\alpha(y) = \alpha(x \ast y) = 0$, and so
\[ 0 = \alpha(x \ast y) = \alpha(x) \ast \alpha(y) = \alpha(x) \ast 0 = \alpha(x). \]
This implies $x \in \text{Ker}(\alpha)$. This completes the proof.

Proposition 3.12. Let $\alpha$ be a right fixed map of $X$. If $\alpha$ is one-to-one, then $\text{Ker}(\alpha) = 0$.

Proof. Suppose that $\alpha$ is one-to-one and $x \in \text{Ker}(\alpha)$. Then $\alpha(x) = 0 = \alpha(0)$, and thus $x = 0$, i.e., $\text{Ker}(\alpha) = \{0\}$.

Denote by $RF(X)$ the set of all right fixes maps of $X$. Let $\otimes$ be a binary operation on $RF(X)$ defined by
\[ (\alpha \otimes \beta)(x) = \alpha(x) \ast \beta(x) \]
for all $\alpha, \beta \in RF(X)$ and $x \in X$. 
Proposition 3.13. Let $X$ be an $Q$-algebra. Then $(RF(X), \otimes)$ is an $Q$-algebra of $X$.

Proof. Let $\alpha \in RF(X)$ and $x \in X$. Then we get $(\alpha \otimes \alpha)(x) = \alpha(x) \ast \alpha(x) = 0$, which proves (Q1). Similarly, we can prove (Q2) and (Q3). □

Let $IRF(X)$ denote the set of all idempotent right fixed maps of $X$.

Theorem 3.14. Let $X$ be a self-distributive $Q$-algebra of $X$ and $\alpha, \beta \in IRF(X)$. Then we have

(1) $\alpha \otimes \beta \in RF(X)$,

(2) If $\alpha(\beta(x)) = \beta(\alpha(x))$ for all $x \in X$, then $\alpha \otimes \beta \in IRF(X)$.

Proof. (1) For every $x, y \in X$, we get

\[(\alpha \otimes \beta)(x \ast y) = \alpha(x \ast y) \ast \beta(x \ast y) = (\alpha(x) \ast y) \ast (\beta(y) \ast y) = (\alpha(x) \ast \beta(x)) \ast y = (\alpha \otimes \beta)(x \ast y),\]

and so $\alpha \otimes \beta \in RF(X)$.

(2) Suppose that $\alpha(\beta(x)) = \beta(\alpha(x))$ for all $x \in X$. Then

\[(\alpha \otimes \beta)((\alpha \otimes \beta)(x)) = (\alpha \otimes \beta)(\alpha(x) \ast \beta(x)) = \alpha(\alpha(x) \ast \beta(x)) \ast \beta(\alpha(x) \ast \beta(x)) = (\alpha(\alpha(x)) \ast \beta(x)) \ast (\beta(\alpha(x)) \ast \beta(x)) = (\alpha(x) \ast \beta(x)) \ast (\alpha(\beta(x)) \ast (\beta(x))) = (\alpha(x) \ast \beta(x)) \ast (\beta(x) \ast \beta(x)) = (\alpha(x) \ast \beta(x)) \ast \alpha(0) = (\alpha(x) \ast \beta(x)) \ast 0 = (\alpha \otimes \beta)(x),\]

that is, $\alpha \otimes \beta$ is idempotent. Hence we obtain $(\alpha \otimes \beta) \in IRF(X)$. □

References


Received: August, 2010