On the Left Ideals of Group Algebra
of the Affine Group

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Abstract

From the fact that the affine group $G = \mathbb{R} \rtimes \rho \mathbb{R}_+^*$ has only two equivalence classes of unitary irreducible representations, as such it is difficult to do its non commutative Fourier analysis. The aim of this paper is computing all left ideal of the group algebra $L^1(G)$, without using the theory of representations. Moreover we generalize the Fourier transform: Motivated by some recent work [4, 6], in order to establish the Plancheral formula

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1 Introduction.

1.1. Let $A$ be the affine group which consisting of all matrices of the form

$$
\begin{pmatrix}
    x & y \\
    0 & 1
\end{pmatrix}
\quad x > 0, \ y \in \mathbb{R}
$$

(1)

Let $\mathbb{R}_+^* = \{x \in \mathbb{R}; \ x > 0\}$ be the multiplicative group of all positive real numbers. Let $G = \mathbb{R} \rtimes \rho \mathbb{R}_+^*$ be the group of the semi-direct product of $\mathbb{R}$ and $\mathbb{R}_+^*$, via the group homomorphism $\rho : \mathbb{R}_+^* \longrightarrow Aut(\mathbb{R})$ defined by:

$$
\rho(x)(y) = x \cdot y
$$

(2)
for any \( x \in \mathbb{R}_+^* \) and \( y \in \mathbb{R} \), where \( \text{Aut}(\mathbb{R}) \) is the group of all automorphisms of \( \mathbb{R} \). So by [14, P.238 – 240] or [4], the group \( H \) can be identified with the group \( G = \mathbb{R} \ltimes_\rho \mathbb{R}_+^* \). The multiplication of two elements \( X = (y, x) \) and \( Y = (y', x') \) in \( G \) is given by

\[
X \cdot Y = (y, x) \cdot (y', x') = (y + xy', x \cdot x')
\]

and the inverse of an element \( X = (y, x) \) in \( G \) is:

\[
X^{-1} = (y, x)^{-1} = (-x^{-1}y, x^{-1})
\]

1.2. If \( M \) is an analytic Lie group, we denote by \( L^1(M) \) the Banach algebra that consists of all complex valued functions on the group \( M \), which are integrable with respect to the right Haar measure of \( M \) and multiplication is defined by convolution on \( M \). Denote by \( C^\infty(G), \mathcal{D}(G), \mathcal{D}'(G), \mathcal{E}'(G) \) respectively the space of \( C^\infty \)-functions, \( C^\infty \)-functions with compact support, distributions and distributions with compact support. Let \( \mathcal{U} \) be the complexified universal enveloping algebra of the real Lie algebra \( g \) of \( G \), which is canonically isomorphic onto the algebra of all distributions on \( G \) supported by \( \{(0, 1)\} \) the identity element of \( G \). For any \( u \in \mathcal{U} \), one can define a differential operator \( P_u \) on \( G \) as follows:

\[
P_u f(X) = u \ast f(X) = \int_G f(Y^{-1}X)u(Y) \, dY
\]

for any \( f \in C^\infty(G) \), where \( X = (y, x), Y = (y', x') \) and \( dY = dy' \frac{dx'}{x} \) is the right Haar measure on \( G \). The mapping \( u \mapsto P_u \) is an algebra isomorphism of \( \mathcal{U} \) onto the algebra of all right invariant differential operators on \( G \). For more details see [5, 9].

1.3. Let \( A = \mathbb{R} \times \mathbb{R}_+^* \) be the group of the direct product of \( \mathbb{R} \) by \( \mathbb{R}_+^* \), we denote also by \( \mathcal{U} \) its enveloping algebra. For every \( u \in \mathcal{U} \), we can associate a differential operator \( Q_u \) on \( A \) as follows:

\[
Q_u f(X) = u \ast_c f(X) = \int_A f(X - Y)u(Y) \, dY
\]

for any \( f \in C^\infty(A) \), \( X \in A \) and \( Y \in A \), where \( \ast_c \) signifies the commutative convolution product on the group \( A \) and \( dY = dy \frac{dx}{x} \) is the Lebesgue-Haar
measure on \( A \). The mapping \( u \mapsto Q_u \) is an algebra isomorphism of \( \mathcal{U} \) onto the algebra of all invariant differential operators on \( A \), which are nothing but the algebra of differential operators with constant coefficients. In this paper the following interesting results will be obtained

**I- Classification of the left ideals of group algebra** \( L^1(G) \) and \( \mathcal{U} \), theorem 2.1.

**II- Establish Plancherel formula theorem** theorem 3.1.

## 2 Ideals of Group algebra \( L^1(G) \).

Let \( K = \mathbb{R} \times \mathbb{R}^*_+ \times \mathbb{R}^*_+ \) be the group of the mixed product \( \mathbb{R} \times \mathbb{R}^*_+ \) and \( \mathbb{R}^*_+ \), with multiplication

\[
X \cdot Y = (z, y, x)(z', y', x') = (z + xz', yy', xx')
\]

for all \((z, y, x) \in K \) and \((z', y', x') \in K \). The inverse of an element \( X = (z, y, x) \) in \( K \) is given by:

\[
X^{-1} = (z, y, x)^{-1} = (-x^{-1}z, y^{-1}, x^{-1})
\]

In this case, we can identify \( G \) with the closed subgroup \( \mathbb{R} \times \{1\} \times \rho \mathbb{R}^*_+ \) if we denote by \( A \) the commutative group \( \mathbb{R} \times \mathbb{R}^*_+ \), which is the direct product of \( \mathbb{R} \) by \( \mathbb{R}^*_+ \), then \( A \) can be identified with subgroup \( \mathbb{R} \times \mathbb{R}^*_+ \times \{1\} \).

**Definition 2.1:** For every \( f \) on \( G \), one can define a function \( \tilde{f} \) as follows:

\[
\tilde{f}(z, y, x) = f(yz, yx)
\]

for all \((z, y, x) \in K \).

**Remark 2.1.** The function \( \tilde{f} \) is invariant in the following sense:

\[
\tilde{f}(kz, yk^{-1}, xk) = \tilde{f}(z, y, x)
\]

for any \((z, y, x) \in K \) and \( k \in \mathbb{R}^*_+ \). So every function \( \psi(z, x) \) on \( G \) extends uniquely as an invariant function \( \psi(z, y, x) \) on \( K \).

**Definition 2.2.** For every \( u \in L^1(G) \) or \( u \in L^1(A) \) one can define two convolutions product on the group \( K \) by:

\[
(i) \quad u * F((z, y, x)) = \int_G F\left((a, b)^{-1}(z, y, x)\right) u(b, a) \frac{da}{a} db
\]

\[
= \int_G F\left[-a^{-1}(z - b), y, xa^{-1}\right] u(b, a) \frac{da}{a} db
\]
(ii) \( u \ast_c F((z, y, x)) = \int_A F((z - b, y, xa^{-1})) u(b, a)\frac{da}{a} db \) \hspace{1cm} (11)

for any \( F \in L^1(K) \), where \( \frac{da}{a} db \) is the right Haar measure on \( G \), \( \ast \) is the convolution product on \( G \) and \( \ast_c \) is the convolution product on \( A \).

**Lemma 2.1.** For every function \( F \in L^1(G) \) and for every \( u \in L^1(G) \), we have

\[ u \ast \tilde{F}(z, y, x) = u \ast \tilde{F}(z, y, x) \] \hspace{1cm} (12)

for every \((z, y; x) \in K\), where \( \ast \) signifies the convolution product on \( G \) and \( \ast_c \) signifies the convolution product on \( A \).

**Proof:** In fact we have

\[
\begin{align*}
    u \ast \tilde{F}(z, y, x) &= \int_G \tilde{F}[(b, a)^{-1}(z, y, x)] u(b, a) db \frac{da}{a} \\
    &= \int_G \tilde{F}[-a^{-1}b, a^{-1})(z, y, x)] u(b, a) db \frac{da}{a} = \int_G \tilde{F}(a^{-1}(z - b), y, xa^{-1})u(b, a) db \frac{da}{a} \\
    &= \int_A \tilde{F}(z - b, ya^{-1}, x) u(b, a) db \frac{da}{a} = u \ast_c \tilde{F}(z, y, x)
\end{align*}
\] \hspace{1cm} (13)

**Proposition 2.1.** The mapping \( \gamma \) from \( L^1(G)|_B \) to \( L^1(G)|_G \) defined by

\[ \tilde{F}|_B (x, y, 0) \rightarrow \gamma(\tilde{F}|_B)(x, 0, y) = \tilde{F}|_G (x, 0, y) \] \hspace{1cm} (14)

is a topological isomorphism, and

\[ \gamma(u \ast_c \tilde{F}|_B)(x, 0, y) = u \ast \tilde{F}|_G(x, 0, y) \] \hspace{1cm} (15)

where

\[ (u \ast_c \tilde{F}|_B)(x, y, 0) = \int_B \tilde{F}[x - t, y - s, 0]) u(t, s) dt ds \] \hspace{1cm} (16)

\[ \gamma(u \ast \tilde{F}|_G)(x, 0, y) = \int_B \tilde{F}[\rho(-s)(x - t), 0, y - s]) u(t, s) dt ds \]

\[ = u \ast F(x, y) \] \hspace{1cm} (17)
Proof: it is enough to see
\[ \gamma(u \ast \tilde{F}|_B)(x, 0, y) = \int_G \tilde{F}[y, \tau(s)] u(t, \tau(s)) \, dt \, ds \]
\[ = \int_G \tilde{F}[\rho(s)(x, 0, 0)], u \ast \tilde{F}|_G(x, 0, y) \]
for every \( F \in L^1(G) \). The fact that \( \gamma : \tilde{L}^1(G)|_B \rightarrow \tilde{L}^1(G)|_G \)
is topological isomorphism can be deduced immediately from proposition 2.1, and so we have
\[ \tilde{F}|_G(x, 0, y) \rightarrow \gamma^{-1}(\tilde{F}|_G)(x, 0, y) = \tilde{F}|_B(x, y, 0) \]
If \( I \) is a subalgebra of \( L^1(G) \), we denote by \( \tilde{I} \) its image by the mapping \( \sim \). Let \( J = \tilde{I} |_B \). Our main result is:

**Theorem 2.1.** Let \( I \) be a subalgebra of \( L^1(G) \), then the following conditions are equivalents.

(i) \( J = \tilde{I} |_B \) is an ideal in the Banach algebra \( L^1(B) \).

(ii) \( I \) is a left ideal in the Banach algebra \( L^1(G) \).

Proof: (i) implies (ii) Let \( I \) be a subalgebra of the space \( L^1(B) \) such that \( J = \tilde{I} |_B \) is an ideal in \( L^1(B) \), then we have:
\[ u \ast \tilde{I} |_B(x, y, 0) \subseteq \tilde{I} |_B(x, y, 0) \]
for any \( u \in L^1(B) \) and \( (x, y) \in B \), where
\[ u \ast \tilde{I} |_B(x, y, 0) = \left\{ \int_B \tilde{F}|_B \, f \, (x - t, y - \tau(s)), u(t, \tau(s)) \ight\} \]
It shows that
\[ u \ast \tilde{F} \in \tilde{I} |_B(x, y, 0) \]
for any \( \tilde{f} \in \tilde{I} \). Apply equation(15), we get
\[ \gamma(u \ast \tilde{F}|_B)(x, 0, y) = u \ast \tilde{F}(x, 0, y) \in \gamma(I |_B(x, 0, y) = I \]
(ii) implies (i) If $I$ is an ideal in $L^1(G)$, then we get

$$u*I|_{G}(x,0,y) = u*I(x,y) \subseteq \tilde{I}|_{G}(x,0,y) = I(x,y)$$

(26)

where

$$u*I|_{G}(x,0,y) = \left\{ \int_{B} \tilde{f}|_{G} [\rho(-s)(x-t),0,y-s] u(t,s)dt\,ds, \quad f \in I \right\}$$

(27)

Apply now equation (21), we obtain

$$\gamma^{-1}(u*\tilde{f}|_{G})(x,0,y) = u*c|_{B}(x,y,0) \in \gamma^{-1}(u*I|_{G})(x,y,0)$$

$$= u*I|_{B}(x,y,0)$$

(28)

**Corollary 2.1.** Let $I$ be a subalgebra of the space $L^1(G)$ and $\tilde{I}$ its image by the mapping $\sim$ such that $J = \tilde{I}|_{A}$ is an ideal in $L^1(A)$, then the following conditions are verified

(i) $J$ is a maximal ideal in the algebra $L^1(A)$ if and only if $I$ is a maximal left ideal in the algebra $L^1(G)$.

(ii) $J$ is a closed ideal in the algebra $L^1(A)$ if and only if $I$ is a closed left ideal in the algebra $L^1(G)$.

(iii) $J$ is a dense ideal in the algebra $L^1(A)$ if and only if $I$ is a dense left ideal in the algebra $L^1(G)$.

(iv) $J$ is a maximal ideal in the algebra $L^1(A)$ if and only if $I$ is a maximal left ideal in the algebra $L^1(G)$.

The proof of this corollary results immediately from theorem 2.1.

### 3 Fourier Transform and Plancherel Formula..

In the following we introduce the notion of the Fourier transform. To do Fourier transform and Plancherel formula, one needs [4, 6] as guideline. Therefor let $\mathcal{S}(G)$ be the Schwartz space of $G$ which can be considered as the Schwartz space of $\mathcal{S}(\mathbb{R} \times \mathbb{R}_+)$, and let $\mathcal{S}'(G)$ be the space of all tempered distributions on $G$. The action $\rho$ of the group $\mathbb{R}_+$ on $\mathbb{R}$ defines a natural action $\rho$ on the dual group $(\mathbb{R})^*$ of the group $\mathbb{R}$, $((\mathbb{R})^* \simeq \mathbb{R})$, which is given by :

$$\rho(x)\xi = x\xi$$

(29)
Definition 3.1. If \( f \in S(G) \), one can define its Fourier transform \( \mathcal{F}f \) by:

\[
\mathcal{F}f (\xi, \lambda) = \int_G f(y, x) e^{-i\xi y} x^{-i\lambda} dy \frac{dx}{x}
\]  

(30)

for any \( \xi \in \mathbb{R} \), \( \lambda \in \mathbb{R} \) and \( \mu \in \mathbb{R} \). It is clear that \( \mathcal{F}f \in S(\mathbb{R}^2) \) and the mapping \( f \rightarrow \mathcal{F}f \) is isomorphism of the topological vector space \( S(G) \) onto \( S(\mathbb{R}^2) \).

Definition 3.2. If \( f \in S(G) \), we define the Fourier transform of its invariant \( \tilde{f} \) as follows:

\[
\mathcal{F}\tilde{f}(\xi, \lambda, \mu) = \int \int \int_G \tilde{f}(z, y, x) e^{-i\xi z} y^{-i\lambda} x^{-i\mu} dz \frac{dy}{y} \frac{dx}{x}
\]  

(31)

Proposition 3.1. For every \( u \in S(G) \) or \( u \in \mathcal{U} \), and \( f \in S(G) \), we have

\[
\mathcal{F}(u \ast \tilde{f})(\xi, \lambda, \mu) = \mathcal{F}(\tilde{f})(\xi, \lambda, \mu) \mathcal{F}(u)(\xi, \lambda)
\]  

(32)

for any \( \xi \in \mathbb{R} \), \( \mu \in \mathbb{R} \) and \( \lambda \in \mathbb{R} \).

Proof: First, we have

\[
u \ast \tilde{f}(z, y, x) = \int_G \tilde{f}((b, a)^{-1}(z, y, x))u(b, a)db \frac{da}{a}
\]

\[
= \int_G \tilde{f}(a^{-1}(z - b), y, xa^{-1})u(b, a)db \frac{da}{a}
\]

\[
= \int_G \tilde{f}(z - b, ya^{-1}, x)u(b, a)db \frac{da}{a}
\]

\[
u \ast \tilde{f}(z, y, x)
\]  

(33)

Secondly:

\[
\int_{\mathbb{R}^2} \mathcal{F}(u \ast \tilde{f})(\xi, \lambda, \mu) d\mu = \int_{\mathbb{R}^2} \mathcal{F}(u \ast \tilde{f})(\xi, \lambda, \mu) d\mu
\]

\[
\mathcal{F}(\tilde{f})(\xi, 1) \mathcal{F}(u)(\xi, \lambda)
\]  

(34)
Theorem 3.1. (Plancheral’s formula)
For any \( f \in L^1(G) \cap L^2(G) \), we get

\[
\hat{f} \ast \hat{f}(0,1,1) = \int_G |f(y,x)|^2 \, dy \frac{dx}{x} = \int_{\mathbb{R}^2} |\mathcal{F}f(\xi,\lambda)|^2 \, d\xi d\lambda
\]

(35)

where \( f(y,x) = \overline{f(y,x)}^{-1} \)

Proof: In fact, we have

\[
f \ast \hat{f}(0,1,1) = \int_G \hat{f} \left[ (-a^{-1}b, a^{-1})(0,1,1) \right] f(b,a) db \frac{da}{a}
\]

\[
= \int_G \hat{f} (-a^{-1}b, 1, a^{-1}) f(b,a) db \frac{da}{a} = \int_G \hat{f} (-a^{-1}b, a^{-1}) f(b,a) db \frac{da}{a}
\]

\[
= \int_G \overline{f(b,a)} f(b,a) db \frac{da}{a} = \int_G |f(b,a)|^2 \, db \frac{da}{a}
\]

(36)

and by (32), we get

\[
f \ast \hat{f}(0,1,1) = \int_{\mathbb{R}^3} \mathcal{F}(f \ast \hat{f})(\xi,\lambda,\mu) d\xi d\lambda d\mu = \int_{\mathbb{R}^3} \mathcal{F}(f \ast \hat{f})(\xi,\lambda,\mu) d\xi d\lambda d\mu
\]

\[
= \int_{\mathbb{R}^2} \mathcal{F}(\hat{f})(\xi,\lambda,1) \mathcal{F}(f)(\xi,\lambda) \xi d\lambda = \int_{\mathbb{R}^2} \mathcal{F}(\hat{f})(\xi,\lambda) \mathcal{F}(f)(\xi,\lambda) d\xi d\lambda
\]

\[
= \int_{\mathbb{R}^2} |\mathcal{F}(f)(\xi,\lambda)|^2 d\xi d\lambda = \int_G |f(b,a)|^2 \, db \frac{da}{a}
\]

(37)

which is the Plancheral’s formula on \( G \).

The Fourier transform can be extended to an isometry of \( L^2(G) \) onto \( L^2(\mathbb{R}^2) \).

Corollary 3.1. From theorem 3.1. we hold

\[
g \ast \hat{f}(0,1,1) = \int_G \overline{f(b,a)} g(b,a) db \frac{da}{a} = \int_{\mathbb{R}^2} \overline{\mathcal{F}f(\xi,\lambda)} \mathcal{F}g(\xi,\lambda) d\xi d\lambda
\]

(38)

which is the Parseval formula on \( G \).
References


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