Representation of One as the Sum of Unit Fractions

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Abstract

One is expressed as the sum of the reciprocals of a certain set of integers. We give an elegant proof to the fact applying the polynomial theorem and basic calculus.

Mathematics Subject Classification: 11B75, 97I40

Keywords: unit fraction, Diophantine equation, mathematical analysis

1 Introduction

Let us consider the representation of one as the sum of unit fractions. For examples, we can take 2, 3 and 6 for

\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1, \quad (1)
\]

and 3, 4, 4, 8 and 24 for

\[
\frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{24} = 1. \quad (2)
\]

It is well known that any positive rational number can be written as the sum of unit fractions. In the paper, we give a part of solutions to the Diophantine equation

\[
\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1, \quad (3)
\]

where the \(x_j\) are not necessarily distinct integer for \(j = 1, 2, \ldots, n\).
Table 1: Possible combinations of $\alpha$ for $n = 6$

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\alpha_5$</th>
<th>$\alpha_6$</th>
<th>denominator</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6! = 720</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4! \cdot 2 = 48</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3! \cdot 3 = 18</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2! \cdot 2! \cdot 2^2 = 16</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2! \cdot 4 = 8</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2 \cdot 3 = 6</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3! \cdot 2^3 = 48</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2 \cdot 4 = 8</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2! \cdot 3^2 = 18</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>

To explain our result for $n = 6$, we find all possible combinations of

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \in \mathbb{N}^6$$

such that

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 6\alpha_6 = 6,$$

where $\alpha_j \in \mathbb{N} = \{0, 1, 2, \ldots\}$ for $j = 1$ to 6. Next, we take the quantities

$$\prod_{j=1}^{6} \alpha_j!j^{\alpha_j} = \alpha_1!1^{\alpha_1} \cdot \alpha_2!2^{\alpha_2} \cdot \ldots \cdot \alpha_6!6^{\alpha_6}$$

for each possible $\alpha$. Then, we can calculate the sum of reciprocals of the above quantities

$$\frac{1}{720} + \frac{1}{48} + \frac{1}{18} + \frac{1}{16} + \frac{1}{8} + \frac{1}{6} + \frac{1}{5} + \frac{1}{48} + \frac{1}{8} + \frac{1}{18} + \frac{1}{6}$$

which is equal to 1.

The example is generalized to our main result:

**Theorem 1.1.** For any positive integer $n$,

$$\sum_{\alpha \in S_n} \prod_{j=1}^{n} \frac{1}{\alpha_j!j^{\alpha_j}} = 1,$$

where the summation over $S_n$ runs through all possible $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbb{N}^n$ such that

$$\sum_{j=1}^{n} j\alpha_j = n.$$
2 Preliminaries

Lemma 2.1 (Polynomial theorem). Let $n$ and $m$ be positive integers. For any $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$,

$$
(x_1 + x_2 + \cdots + x_n)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^{\alpha}
$$

where $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbb{N}^n$ and the summation runs through all possible $\alpha$ in $\mathbb{N}^n$ such that $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = m$.

Proof. Each coefficient of

$$
x^\alpha = \prod_{j=1}^{n} x_j^{\alpha_j}
$$

in the right-hand side for some $\alpha$ in $\mathbb{N}^n$ with $|\alpha| = m$ is equal to the number of combinations of the products among $x_1, x_2, \ldots, x_n$. \qed

Lemma 2.2. Given a polynomial

$$
f(x) = \sum_{j=0}^{n} a_j x^j.
$$

Then, the $j$th coefficient of $f(x)$ can be expressed by

$$
a_j = \frac{1}{j!} f^{(j)}(0),
$$

where $f^{(j)}$ stands for the $j$th derivative.

Proof. $f$ is infinitely differentiable, since the $j$-times differential function of $x^k$ is

$$
\left( \frac{d}{dx} \right)^j x^k = \frac{k!}{(k-j)!} x^{k-j}
$$

if $j \leq k$, and

$$
\left( \frac{d}{dx} \right)^j x^k = 0
$$

if $j > k$. We have

$$
f^{(j)}(x) = \sum_{k=0}^{n} a_k \left( \frac{d}{dx} \right)^j x^k = \sum_{k=j}^{n} \frac{k!}{(k-j)!} a_k x^{k-j}
$$
for any \( j \) between 0 and \( n \), then
\[
f^{(j)}(0) = j!a_j,
\] (17)
which implies the conclusion of the lemma
\[
a_j = \frac{1}{j!} f^{(j)}(0).
\] (18)

**Lemma 2.3.** Let \( n \) be a positive integer. We put
\[
g(x) = \left( x + \frac{1}{2}x^2 + \cdots + \frac{1}{n}x^n \right)^n,
\] (19)
then \( g^{(n)}(0) = n! \).

**Proof.** Put
\[
g(x) = x^n \left( 1 + \frac{1}{2}x + \cdots + \frac{1}{n}x^{n-1} \right)^n = x^n h(x),
\] (20)
where
\[
h(x) = \left( 1 + \frac{1}{2}x + \cdots + \frac{1}{n}x^{n-1} \right)^n.
\] (21)

Leibniz rule implies
\[
g^{(n)}(x) = \sum_{j=0}^{n} \frac{n!}{(n-j)!j!} \left( \frac{d}{dx} \right)^{n-j} x^n \cdot \left( \frac{d}{dx} \right)^j h(x)
\] (22)
Therefore, we obtain
\[
g^{(n)}(0) = n! h(0) = n!.
\] (23)
3 Proof of the main result

We begin with the relation
\[
\left( x_1 + \frac{1}{2}x_2^2 + \cdots + \frac{1}{n}x_n^n \right)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x_1^{\alpha_1} \left( \frac{1}{2}x_2^2 \right)^{\alpha_2} \cdots \left( \frac{1}{n}x_n^n \right)^{\alpha_n} = m! \sum_{|\alpha|=m} \prod_{j=1}^n \frac{1}{\alpha_j!} x_j^{\alpha_j}.
\]
by Lemma ??.

Putting \( m = n \) and \( x_j = t \) for \( j = 1, 2, \ldots, n \) implies
\[
\left( t + \frac{1}{2}t^2 + \cdots + \frac{1}{n}t^n \right)^n = n! \sum_{|\alpha|=n} \prod_{j=1}^n \frac{1}{\alpha_j!} t^{\alpha_j}.
\]

Compare to the coefficients of \( t^n \) in both side of the identity. We obtain \( n! \) from the left-hand side of the identity by Lemma ?? and Lemma ??.

On the other hand, the coefficient of \( t^n \) in the right-hand side is the sum of all terms with \( t^n \), which is written by
\[
\sum_{\alpha \in S_n} \prod_{j=1}^n \frac{1}{\alpha_j!} t^{\alpha_j}.
\]
where the summation runs through all possible \( \alpha \) in \( S_n \) defined by
\[
S_n = \{ \alpha \in \mathbb{N}^n ; \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n \}.
\]

Hence, we obtain
\[
\sum_{\alpha \in S_n} \prod_{j=1}^n \frac{1}{\alpha_j!} t^{\alpha_j} = 1
\]
which completes the proof of our main result.

4 Concluding Remarks

In this paper, a part of reciprocal bases of one is investigated from analytic point of view. In particular, the polynomial theorem and the multi-index analysis play an important role in the proof. Although these are not all of solutions to the Diophantine equation, one is presented as the sum of the reciprocal numbers of a certain set of integers.
References


Received: August, 2010