On the Component Number of Links Corresponding to Sierpiński Graphs

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Abstract

It is well known that there is a one-to-one correspondence between signed plane graphs and link diagrams via the medial construction. In this paper, we determine the number of components of links corresponding to Sierpiński graphs. As a by-product, the parity property of the number of spanning trees of a Sierpiński graph is derived.

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1 Introduction

It is well known that there is a one-to-one correspondence between signed plane graphs and link diagrams via the medial construction [1]. This correspondence was once used to construct a table of link diagrams of all links. In the late 1980s, via this correspondence Kauffman built a relation between the Jones polynomial in knot theory and the Tutte polynomial in graph theory [5] [6].

It is obvious that the component number of the link diagram corresponding to a signed plane graph is independent of the signs of the plane graph. Moreover, it does not depend on the embedding of the plane graph, i.e. link diagrams corresponding to different embeddings of a planar graph have the same component numbers, see [9]. Let $G$ be a planar graph, and $D(G)$ be the link diagram corresponding to one planar embedding of $G$ (signs of edges of $G$ can be chosen arbitrarily). We shall denote by $\mu(D(G))$ the number of components of the link diagram $D(G)$.

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Determining component number of links formed from planar graphs may be one of the first problems in studying links by using graphs. By now, there have been several studies in this aspect, see [7], [9], [8], [2], [4].

We point out that the number of components of the link diagram corresponding to a plane graph has been studied in graph theory for at least forty years, appearing as the number of left-right paths of a plane graph [10]. It also appears in the form of the number of straight-ahead walks of the medial graph of the plane graph in the literature. An algebraic characterization of graphs with link component number equal to 1 was given by a theorem of Shank in 1975 (see Theorem 4 in [10]), stating that:

**Theorem 1.1** Let $G$ be a connected plane graph. Then $G$ corresponds to a knot diagram, i.e. $\mu(D(G)) = 1$ if and only if the spanning tree number $\tau(G)$ of $G$ is odd.

The Sierpiński graph $S_n$ is the discrete analoga of the well known Sierpiński gasket in dynamical systems and probability. A formula for the number of spanning trees of Sierpiński graphs is derived in [12] and it is given by

$$\tau(S_n) = \frac{4\sqrt{3}}{20} \left(\frac{5}{3}\right)^{-(n-1)/2} \left(\sqrt{540}\right)^{3n-1}. \quad (1)$$

Note that it is not easy to judge the parity of the right side of Equation (1). Moreover, the parity is only useful to judge whether the corresponding link diagram is a knot diagram or not. This motivates us to study the component number of the links formed from Sierpiński graphs. In section 2, we give the definition of Sierpiński graphs. In section 3, we determine the component number of links corresponding to Sierpiński graphs, and as a by-product, the parity property of the number of spanning trees of a Sierpiński graph is derived.

## 2 Sierpiński graphs

**Definition 2.1** The Sierpiński graph $S_n$ can be defined recursively as follows:

1. $S_1$ is the triangle, i.e. the graph $K_3$.

2. $S_{n+1}$ is formed from three copies of $S_n$ (which we will refer to as the top, bottom left and bottom right parts of $S_{n+1}$ and denote by $S_{n+1,T}$, $S_{n+1,L}$ and $S_{n+1,R}$, respectively) by attaching three pairs of vertices:
   \[
   \{S_{n+1,T,L}, S_{n+1,L,T}\}, \{S_{n+1,T,R}, S_{n+1,R,T}\} \text{ and } \{S_{n+1,L,R}, S_{n+1,R,L}\}.
   \]
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Figure 1: The construction of $S_{n+1}$.

See Figure 1 for the recursive construction of Sierpiński graphs. The three degree two vertices $\{S_{n,T,T}, S_{n,L,L}, S_{n,R,R}\}$ of $S_n$ will be called “outside vertices” and the three vertices $\{S_{n,T,L}, S_{n,L,R}, S_{n,T,R}\}$ of $S_n$ will be called “middle vertices”. As examples, Sierpiński graphs $S_2, S_3$ and $S_4$ are portrayed in Figure 2.

There have been some studies on Sierpiński graphs in recent years. For example, in [11], the authors obtained some results on cycle structure, domination number and pebbling number of Sierpiński graphs. In [3], the authors considered the vertex coloring, edge-coloring and total-coloring of Sierpiński graphs.

Figure 2: $S_2, S_3$ and $S_4$. 
3 Main Results

Throughout this section, we shall use thin lines to draw graphs and thick (solid or dashed) lines to draw link diagrams.

**Lemma 3.1** Let $S_n$ ($n \geq 2$) be a Sierpiński graph. Then, in $D(S_n)$ the three small segments of arcs outside the three “outside vertices” of $S_n$ belong to one link component and the three small segments of arcs outside the three “middle vertices” of $S_n$ belong to another link component as shown in Figure 3.

![Figure 3: Two link components of $D(S_n)$](#)

**Proof.** We prove the lemma by induction on $n$. As shown in Figure 4 (below), the lemma holds for $n = 2$.

![Figure 4: $S_1$ (above) and $S_2$ (below)](##)
Now we suppose that the lemma holds for \( n = k \) and now consider the case \( n = k + 1 \). By induction hypothesis, the three small segments of arcs outside \( S_{k,T,T}, S_{k,L,L} \) and \( S_{k,R,R} \) of \( S_k \) belong to a same link component. Now we use three copies \( S_{k+1,T}, S_{k+1,L}, S_{k+1,R} \) of \( S_k \) to construct \( S_{k+1} \). The three segments of arcs drawn with dashed lines as shown in Figure 5 will be changed to form the component link outside the three “middle vertices” of \( S_{k+1} \). The other six solid lines will be changed to form the component link outside the three “outside vertices” of \( S_{k+1} \). This completes the proof of the lemma.

\[ \text{Figure 5: } S_k \rightarrow S_{k+1}. \]

**Theorem 3.2** Let \( n \) be a positive integer. Then

\[
\mu(D(S_n)) = \frac{3^{n-1} - 1}{2} + 1.
\]

**Proof.** We shall prove the theorem by induction on \( n \). If \( n = 1 \), \( S_1 \) is the triangle. Note that triangle corresponds to trefoil knot projection as shown in Figure 4 (above). Now we suppose the theorem holds for \( n = k \), where \( k \geq 1 \). Recall that \( S_{k+1} \) can be obtained from three copies of \( S_k \) as shown in Figure 1. From the proof the lemma 3.1, we know that the three components outside the three “outside vertices” of three copies of \( S_k \) becomes two link components of
the link diagram corresponding to $S_{k+1}$ and other components are unchanged in the construction from $S_k$ to $S_{k+1}$. Hence we have

$$\mu(D(S_{k+1})) = 3(\mu(D(S_k)) - 1) + 2$$

Thus

$$\mu(D(S_{k+1})) = 3\mu(D(S_k)) - 1$$

$$= 3 \left( \frac{3^{k-1} - 1}{2} + 1 \right) - 1$$

$$= \frac{3^k - 1}{2} + 1.$$

This completes the proof of the theorem.

**Example 3.3** The Sierpiński graph $S_4$ and its corresponding link diagram $D(S_4)$ are shown in Figure 6. $\mu(D(S_4)) = 14$, which matches Theorem 3.2.

![Figure 6: $S_4$ and $D(S_4)$](image)

Combining Theorems 1.1, 3.2 and Equation (1), we obtain

**Corollary 3.4** $\sqrt[4]{\frac{3}{20}} \left( \frac{5}{3} \right)^{-n/2} (\sqrt{540})^3$ is an even integer for each $n \geq 1$.

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