Absorbing Maps and Fixed Point Theorem in Fuzzy Metric Spaces

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Abstract

In this paper, we prove a common fixed point theorem for six mappings using absorbing maps in \( \epsilon \)-chainable fuzzy metric space. Our paper extends the results of Cho et al [1]

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1 Introduction

In 1965 Zadeh [14] introduced the notion of fuzzy sets. Later many authors have extensively developed the theory of fuzzy sets and application. The idea of fuzzy metric space introduced by Kramosil and Michalek [7] was modified by George and Veeramani [5]. Singh et al [11] introduced the notion of semi-compatible maps in fuzzy metric spaces, and obtain fixed point theorems in complete fuzzy metric space in the sense of Grabiec [4]. In [13] Vasuki introduce the concept of \( \epsilon \)-weakly commuting map, and prove a fixed point theorem in fuzzy metric space. In Cho et al [1] introduce the concept of \( \epsilon \)-chainable fuzzy metric space and obtain common fixed point theorems for four weakly compatible mappings of \( \epsilon \)-chainable fuzzy metric space. Recently we [10] introduced the concept of absorbing maps in metric space and prove the
common fixed point theorem in this spaces; we [10] observe that the new notion of absorbing map is neither a subclass of compatible maps nor a subclass of non-compatible maps. In [8] we apply the notion of absorbing maps, in fuzzy metric spaces and prove a common fixed point theorem in this spaces. In this paper, we prove a common fixed point theorem for six mappings using absorbing maps with $\epsilon$-chainable fuzzy metric space. Our paper extends the results of Cho et al [1].

2 Preliminaries

In this section we recall some definitions and known results in fuzzy metric space.

**Definition 2.1** A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-norm if $([0, 1], *)$ is an abelian topological monoid with 1 such that $a*b \leq c*d$, whenever $a \leq c$, $b \leq d$ for all $a, b, c, d \in [0, 1]$.

**Definition 2.2** The $3$-tuple $(X, M, \star)$ is called a fuzzy metric space if $X$ is an arbitrary non-empty set, $\star$ is a continuous $t$-norm and $M$ is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$.

1. (FM1) $M(x, y, 0) > 0$ ;
2. (FM2) $M(x, y, t) = 1$ for all $t > 0$, iff $x = y$ ;
3. (FM3) $M(x, y, t) = M(y, x, t)$ ;
4. (FM4) $M(x, y, t) \star M(y, z, t) \geq M(x, z, t + s)$ ;
5. (FM5) $M(x, y, .): [0, \infty) \rightarrow [0, 1]$ is left continuous.

Let $(X, d)$ be a metric space. Define $a*b = ab$ (or $a*b = \min[a, b]$) and for all $x, y \in X$ and $t > 0$, $M(x, y, t) = \frac{t}{t+d(x,y)}$. Then $M(X, M, \star)$ is a fuzzy metric space. We call this fuzzy metric $M$ induced by the metric $d$ the standard fuzzy metric.

**Definition 2.3** A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \star)$ is said to be convergent to a point $x \in X$ if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in N$ such that $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1 - \epsilon$ for each $t > 0$. A sequence $\{x_n\}$ in a fuzzy metric space $M(X, M, \star)$ is called Cauchy if $\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1 - \epsilon$ for each $t > 0$ and $p > 0$. Fuzzy metric space $M(X, M, \star)$ is said to complete if every Cauchy sequence in $X$ converge to a point in $X$.

**Definition 2.4** A pair $(A, B)$ of self maps of a fuzzy metric space $(X, M, \star)$ is said to be reciprocal continuous if $\lim_{n \rightarrow \infty} ABx_n = Ax$ and $\lim_{n \rightarrow \infty} BAX_n = Bx$, whenever there exists a sequence $x \in X$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = x$ for some $x \in X$. If $A$ and $B$ are both continuous then they are obviously reciprocally continuous but not converse need not be true.
Definition 2.5 A pair \((A, B)\) of self-maps of a fuzzy metric space \((X, M, \ast)\) is said to be semi-compatible if \(\lim_{n \to \infty} ABx_n = Bx\) whenever there exist a sequence \(x_n \in X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x\) for some \(x \in X\).

Lemma 2.6: If for all \(x, y \in X\), \(t > 0\) and \(0 < k < 1\), \(M(x, y, kt) \geq M(x, y, t)\), then \(x = y\).

Lemma 2.7 (Grabiec [4]): \(M(x, y, \cdot)\) is non-decreasing for all \(x, y \in X\).

Definition 2.8: Let \(A\) and \(B\) be two self-maps on a fuzzy metric space \((X, M, \ast)\) then \(A\) is called \(B\)-absorbing if there exists a positive \(R > 0\) such that \(M(Bx, BAx, t) \geq M(Bx, Ax, \frac{t}{R})\) for all \(x \in X\). Similarly \(B\) is called \(A\)-absorbing if there exists a positive integer \(R > 0\) such that \(M(Ax, ABx, t) \geq M(Ax, Bx, \frac{t}{R})\) for all \(x \in X\).

Example 2.9 (George and Veeramani [5]): Let \((X, d)\) be usual metric space where \(X = [2, 20]\) and \(M\) be the usual fuzzy metric on \((X, M, \ast)\) where \(\ast = t_{\min}\) be the induced fuzzy metric space with \(M(x, y, t) = \frac{t}{t + d(x, y)}\) and \(M(x, y, 0) = 0\) for \(x, y \in X, t > 0\). We define

\[
Ax = \begin{cases} 
6 & \text{if } 2 \leq x \leq 5; \\
10 & \text{if } x > 6 \\
\frac{x-1}{2} & \text{if } x \in (5, 6)
\end{cases}
\]

\[
Bx = \begin{cases} 
2 & \text{if } 2 \leq x \leq 5 \\
\frac{x+1}{3} & \text{if } x > 5
\end{cases}
\]

It is easy to see that both \((A, B)\) and \((B, A)\) are not compatible but \(A\) is \(B\)-absorbing and \(B\) is \(A\)-absorbing. [Hint: Choose \(x_n = 5 + \frac{1}{2n}; x \in N\)] See ([8] Ranadive et al).

Example 2.10 (Ranadive et al [8]): If \(X = [0, 1]\) be a metric space and \(d\) and \(M\) are same as above example 2.9. Define \(A, B: X \to X\) by \(Ax = \frac{x}{16}\) and \(Bx = 1 - \frac{4}{3}x\). In this example we can see that \(A\) and \(B\) are compatible pair of maps and \(A\) is \(B\)-absorbing while \(B\) is \(A\)-absorbing [Hint : range of \(A = [0, \frac{1}{16}]\) and range of \(B = [\frac{2}{3}, 1]\)]

Our next example to show that absorbing maps need not commute at their coincidence points, thus the notion of absorbing maps is different from other generalization of commutativity which force the mapping to commute at coincidence points.
Example 2.11: Let \( X = [0, 1] \) be a metric space and \( d \) and \( M \) are same as in above example 2.9. Define \( A, B : X \to X \) by
\[
Ax = \begin{cases} 
1 & \text{for } x \neq 1 \\
0 & \text{for } x = 1
\end{cases}
\]
and \( Bx = 1 \) for \( x \in X \). Then the maps \( A \) and \( B \)-absorbing for any \( R > 1 \) but the pair of maps \((A, B)\) do not commute at their coincidence point \( x = 0 \).

Cho et al [1], has been established the following result:

Theorem 2.12 Let \((X, M, *)\) be a complete \(\epsilon\)-chainable fuzzy metric space and let \(A, B, S\) and \(T\) be self mappings of \(X\) satisfying the following conditions:

1. \(AX \subset TX \) and \(BX \subset SX\);
2. \(A \) and \(S\) are continuous;
3. the pairs \([A, S]\) and \([B, T]\) are weakly compatible;
4. there exists \(q \in (0, 1)\) such that
\[
M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Ax, Ty, t)
\]
for every \(x, y \in X\) and \(t > 0\). Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

3 Main Results

In this section we prove common fixed point theorem using absorbing maps for six mappings.

Theorem 3.1 Let \(A, B, S, T, L\) and \(M\) be a complete \(\epsilon\)-chainable fuzzy metric space \((X, M, *)\) with continuous \(t\)-norm satisfying the conditions.

1. \(L(X) \subseteq ST(X), M(X) \subseteq AB(X)\);
2. \(AB = BA, ST = TS, LB = BL, MT = TM\);
3. \(M\) is \(ST\)-absorbing;
4. there exists \(k \in (0, 1)\) such that
\[
M(Lx, My, kt) \geq \min\{M(ABx, My, (2 - \alpha)t), M(ABx, STy, t), M(ABx, Lx, t), M(STy, My, t)\}
\]
for every \(x, y \in X, \alpha \in (0, 2)\) and \(t > 0\). If \(L, AB\) is reciprocally continuous, semi-compatible maps. Then \(A, B, S, T, L\) and \(M\) have a unique common fixed point in \(X\).
Proof: Let \( x_0 \in X \) then from (1) there exists \( x_1, x_2 \in X \) such that \( Lx_0 = STx_1 = y_0 \) and \( Mx_1 = ABx_2 = y_1 \). In general we can find a sequence \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that \( Lx_{2n} = STx_{2n+1} = y_{2n} \) and \( Mx_{2n+1} = ABx_{2n+2} = y_{2n+1} \) for \( n = 0, 1, 2, \ldots \) putting \( x = x_{2n+2} \) and \( y = x_{2n+1} \) for all \( t > 0 \) and \( \alpha = 1 - q \) with \( q \in (0, 1) \) in (4), we have

\[
M(y_{2n+1}, y_{2n+2}, kt) = M(Lx_{2n+2}, Mx_{2n+1}, kt)
\]

\[
\geq \min \{M(ABx_{2n+2}, Mx_{2n+1}, (2 - (1 - q)t)), M(ABx_{2n+2}, STx_{2n+1}, t)
\]

\[
M(ABx_{2n+2}, Lx_{2n+2}, t), M(STx_{2n+1}, Mx_{2n+1}, t)\}
\]

\[
\geq \min \{M(y_{2n+1}, y_{2n+2}, (1 + q)t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+2}, t),
\]

\[
M(y_{2n+1}, y_{2n+2}, t)\},
\]

\[
= \min \{1, M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+2}, t)\}
\]

\[
M(y_{2n+1}, y_{2n+2}, kt) \geq \min M(y_{2n+1}, y_{2n+2}, t).
\]

Again \( x = x_{2n+2} \) and \( y = x_{2n+3} \) with \( \alpha = 1 - q \) with \( q \in (0, 1) \) in (4), we have

\[
M(y_{2n+2}, y_{2n+3}, kt) = M(Lx_{2n+2}, Mx_{2n+3}, kt)
\]

\[
\geq \min \{M(ABx_{2n+2}, Mx_{2n+3}, (1+q)t)), M(ABx_{2n+2}, STx_{2n+3}, t), M(ABx_{2n+2}, Lx_{2n+2}, t),
\]

\[
M(STx_{2n+3}, Mx_{2n+3}, t)\}
\]

\[
\geq \min \{M(y_{2n+1}, y_{2n+2}, (1 + q)t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+2}, t),
\]

\[
M(y_{2n+1}, y_{2n+2}, t)\},
\]

\[
= \min \{M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n+3}, qt), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+2}, t),
\]

\[
M(y_{2n+2}, y_{2n+3}, t)\}.
\]

As t-norm is continuous, letting \( q \to 1 \) we have,

\[
M(y_{2n+2}, y_{2n+3}, kt) \geq \min \{M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n+3}, t)\}
\]

Hence

\[
M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t).
\]

Therefore for all \( n \); we have

\[
M(y_n, y_{n+1}, t) \geq M(y_n, y_{n-1}, t/k) \geq M(y_n, y_{n-1}, t/k^2) \geq \ldots \geq M(y_n, y_{n-1}, t/k^n)
\]

\[
\to 1 \text{ as } n \to \infty.
\]

for any \( t > 0 \). For each \( \epsilon > 0 \) and each \( t > 0 \), we can choose \( n_0 \in N \) such that \( M(y_n, y_{n+1}, t) > 1 - \epsilon \) for all \( n > n_0 \). For \( m, n \in N \), we suppose \( m \geq n \). Then we have that

\[
M(y_n, y_m, t) \geq M(y_n, y_{n+1}, t/m - n), M(y_{n+1}, y_{n+2}, t/m - n), \ldots
\]
\[ M(y_{m-1}, y_m, t/m - n) > (1 - \varepsilon) \ast (1 - \varepsilon) \ast (1 - \varepsilon) \ast \ldots \ast (1 - \varepsilon) \geq (1 - \varepsilon). \] Hence \( \{y_n\} \) is a Cauchy sequence in \( X \); that is \( y_n \to z \) in \( X \); so its subsequences \( Lx_{2n}, STx_{2n+1}, ABx_{2n}, Mx_{2n+1} \) also converges to \( z \). Since \( X \) is \( \epsilon \)-chainable, there exists \( \epsilon \)-chain from \( x_n \) to \( x_{n+1} \), that is, there exists a finite sequence \( x_n = y_1, y_2, \ldots, y_l = x_{n+1} \) such that \( M(y_i, y_{i-1}, t) > 1 - \varepsilon \) for all \( t > 0 \) and \( i = 1, 2, \ldots, l \). Thus we have \( M(x_n, x_{n+1}, t) > M(y_1, y_2, t/l) \ast M(y_2, y_3, t/l) \ast \ldots \ast M(y_{l-1}, y_l, t/l) > (1 - \varepsilon) \ast (1 - \varepsilon) \ast \ldots \ast (1 - \varepsilon) \geq (1 - \varepsilon) \), and so \( \{x_n\} \) is a Cauchy sequence in \( X \) and hence there exists \( z \in X \) such that \( x_n \to z \). Since the pair of \( (L, AB) \) is reciprocal continuous; we have \( \lim_{n \to \infty} L(AB)x_{2n} \to Lz \) and \( \lim_{n \to \infty} AB(L)x_{2n} \to ABz \) and the semi-compatibility of \( (L, AB) \) which gives \( \lim_{n \to \infty} AB(L)x_{2n} \to ABz \), therefore \( Lz = ABz \). We claim

\[ Lz = ABz = z. \]

**Step 1:** Putting \( x = z \) and \( y = x_{2n+1} \) with \( \alpha = 1 \) in (4), we have

\[
M(Lz, Mx_{2n+1}, kt) \geq \min\{ M(ABz, Mx_{2n+1}, t), M(ABz, STx_{2n+1}, t), M(ABz, Lz, t), \\
M(STx_{2n+1}, Mx_{2n+1}, t) \}
\]

Letting \( n \to \infty \); we have

\[
M(Lz, z, kt) \geq \min\{ M(Lz, z, t), M(Lz, z, t), M(Lz, Lz, t), M(z, z, t) \}
\]

i.e.

\[ z = Lz = ABz. \]

**Step 2:** Putting \( x = Bz, y = x_{2n+1} \) with \( \alpha = 1 \) in (4), we have

\[
M(L(Bz), Mx_{2n+1}, kt) \geq \min\{ M(AB(Bz), Mx_{2n+1}, t), M(AB(Bz), STx_{2n+1}, t), \\
M(AB(Bz), L(Bz), t), M(STx_{2n+1}, Mx_{2n+1}, t) \}
\]

Since \( LB = BL, AB = BA \), so \( L(Bz) = B(Lz) = Bz \) and \( AB(Bz) = B(ABz) = Bz \) Letting \( n \to \infty \); we have

\[
M(Bz, z, kt) \geq \min\{ M(Bz, z, t), M(Bz, z, t), M(Bz, z, t), M(z, z, t) \}
\]

i.e.

\[ M(Bz, z, kt) \geq M(Bz, z, t) \]

Therefore

\[ Az = Bz = Lz = z. \]

**Step 3:** Since \( L(X) \subseteq ST(X) \), there exists \( u \in X \), such that \( z = Lz = STu \).

Putting \( x = x_{2n}, y = u \) with \( \alpha = 1 \) in (4), we have

\[
M(Lx_{2n}, Mu, kt) \geq \min\{ M(ABx_{2n}, Mu, t), M(ABx_{2n}, STu, t), M(ABx_{2n}, Lx_{2n}, t) \},
\]
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Let $n \to \infty$; we have

$$M(z, Mu, kt) \geq \min \{M(z, Mu, t), M(z, z, t), M(z, Mu, t)\}$$

i.e.

$$M(z, Mu, kt) \geq M(z, Mu, t)$$

Therefore

$$z = Mu = STu.$$ 

Since $M$ is $ST$-absorbing; then

$$M(STu, STMu, kt) \geq M(STu, Mu, t/R) = 1$$

i.e. $STu = STMu \Rightarrow z = STz.$

Step 4: Putting $x = x_{2n}, y = z$ with $\alpha = 1$ in (4), we have

$$M(Lx_{2n}, Mz, kt) \geq \min \{M(ABx_{2n}, Mz, t), M(ABx_{2n}, STz, t), M(ABx_{2n}, Lx_{2n}, t), M(STz, Mz, t)\}$$

Letting $n \to \infty$; we have

$$M(z, Mz, kt) \geq \min \{M(z, Mz, t), M(z, z, t), M(z, Mz, t)\}$$

i.e.$M(z, Mz, kt) \geq M(z, Mz, t)$Therefore

$$z = Mz = STz.$$ 

Step 5 : Putting $x = x_{2n}, y = Tz$ with $\alpha = 1$ in (4), we have

$$M(Lx_{2n}, M(Tz), kt) \geq \min \{M(ABx_{2n}, M(Tz), t), M(ABx_{2n}, ST(Tz), t), M(ABx_{2n}, Lx_{2n}, t), M(ST(Tz), M(Tz), t)\}$$

Since $MT = TM, ST = TS$ therefore $M(Tz) = T(Mz) = Tz, ST(Tz) = T(STz) = Tz$; Letting $n \to \infty$; we have

$$M(z, Tz, kt) \geq \min \{M(z, Tz, t), M(z, z, t), M(z, z, t), M(Tz, Tz, t)\}$$

i.e.$M(z, Tz, kt) \geq M(z, Tz, t)$Therefore

$$z = Tz = Sz = Mz.$$ 

Hence

$$z = Az = Bz = Lz = Sz = Mz = Tz.$$
Uniqueness: Let $w$ be another fixed point of $A, B, L, S, M$ and $T$. Then Putting $x = u, y = w$ with $\alpha = 1$ in (4), we have

\[ M(Lu, Mw, kt) \geq \min \{ M(ABu, Mw, t), M(ABu, STw, t), M(ABu, Lu, t), M(STw, Mw, t) \} \]

\[ \geq \min \{ M(u, w, t), M(u, w, t), M(u, u, t), M(w, w, t) \} \]

Therefore

\[ M(u, w, kt) \geq M(u, w, t). \]

Hence

\[ z = w. \]

**Corollary 3.2**: Let $A, B, S, T, L$ and $M$ be a complete $\epsilon$-chainable fuzzy metric space $(X, M, \ast)$ with continuous $t$-norm satisfying the conditions (1) to (3) of Theorem 3.1 and:

(5) there exists $k \in (0, 1)$ such that

\[ M(Lx, My, kt) \geq \min \{ M(ABx, My, (2 - \alpha)t), M(ABx, STy, t), M(ABx, Lx, t), M(STy, My, t), M(STy, Lx, 2t) \} \]

for every $x, y \in X, \alpha \in (0, 2)$ and $t > 0$. If $L, AB$ is reciprocally continuous, semi-compatible maps. Then $A, b, S, T, L$ and $M$ have a unique common fixed point in $X$.

Proof: we have

\[ M(Lx, My, kt) \geq \min \{ M(ABx, My, (2 - \alpha)t), M(ABx, STy, t), M(ABx, Lx, t) \} \]

\[ \geq \min \{ M(ABx, My, (2 - \alpha)t), M(ABx, STy, t), M(ABx, Lx, t) \} \]

\[ \geq \min \{ M(ABx, My, (2 - \alpha)t), M(STy, My, t), M(STy, ABx, t), M(ABx, Lx, t) \} \]

\[ \geq \min \{ M(ABx, My, (2 - \alpha)t), M(STy, My, t), M(STy, ABx, t), M(ABx, Lx, t) \} \]

and hence Theorem 3.1, $A, B, S, T, L$ and $M$ have a unique common fixed in $X$. Let $B$ and $S$ be the identity mapping on $X$ in Theorem 3.1. Then we get the next results.
Corollary 3.3: Let $A, T, L$ and $M$ be a complete $\epsilon$-chainable fuzzy metric space $(X, M, \ast)$ with continuous t-norm satisfying the conditions.

1. $L(X) \subseteq T(X), M(X) \subseteq A(X)$;
2. $M$ is $T$-absorbing;
3. there exists $k \in (0, 1)$ such that

$$M(Lx, My, kt) \geq \min\{M(Ax, My, (2 - \alpha)t), M(Ax, Ty, t), M(Ax, Lx, t), M(Ty, My, t)\}$$

for every $x, y \in X, \alpha \in (0, 2)$ and $t > 0$. If $\{L, A\}$ is reciprocally continuous, semi-compatible maps. Then $A, T, L$ and $M$ have a unique common fixed point in $X$.

References


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