\textbf{\textit{\textbf{\textit{$\mathcal{X}$-Gorenstein Projective Modules}}}}

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Abstract

In this paper, we generalize the notion of Gorenstein projective modules. Namely, we introduce $\mathcal{X}$-Gorenstein projective modules, where $\mathcal{X}$ is a class of modules that contains all projective modules. We show that the principal results on Gorenstein projective module remain true for the $\mathcal{X}$-Gorenstein projective modules.

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1 Introduction

Throughout this paper, $R$ denotes a non-trivial associative ring with identity, and all modules are left $R$-modules.

In 1967-69, Auslander and Bridger [1, 2] introduced the G-dimension for finitely generated $R$-modules when $R$ is Noetherian, denoted by $G \dim(M)$ where $M$ is a finitely generated $R$-module. As the classical case, the G-dimension of modules is defined in terms of resolutions by modules of G-dimension 0, which are defined as follows:

A finitely generated $R$-module $M$ has G-dimension 0, if:

- $\text{Ext}^m_R(M, R) = 0 = \text{Ext}^m_R(\text{Hom}_R(M, R), R)$ for every $m > 0$; and
• \( M \) is reflexive, that is, the canonical map \( M \to \text{Hom}_R(\text{Hom}_R(M, R), R) \) is an isomorphism.

In [1], Auslander proved that a finitely generated \( R \)-module \( M \) has G-dimension 0 if and only if there exists an exact sequence of finitely generated free \( R \)-modules \( L = \cdots \to L_1 \to L_0 \to L^0 \to L^1 \to \cdots \) such that \( M \cong \text{Im}(L_0 \to L^0) \) and the complex \( \text{Hom}_R(L, R) \) is exact.

In [5, 6], Enochs and Jenda defined, over arbitrary rings, the Gorenstein projective modules as follows:

**Definition 1.1** An \( R \)-module \( M \) is said to be Gorenstein projective, if there exists an exact sequence of projective \( R \)-modules 
\[
P = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots
\]

such that \( M \cong \text{Im}(P_0 \to P^0) \) and such that \( \text{Hom}_R(\cdot, Q) \) leaves the sequence \( P \) exact whenever \( Q \) is a projective \( R \)-module.

The exact sequence \( P \) is called a complete projective resolution.

Over a Noetherian ring \( R \), Avramov, Buchweitz, Martsinkovsky, and Reiten proved that a finitely generated \( R \)-module \( M \) is Gorenstein projective if and only if \( \text{G} - \dim(M) = 0 \) (see [4, Theorem 4.2.6] and [4, notes p. 99]). So, the notion of Gorenstein projective modules is an extension of the notion of modules of G-dimension 0. Furthermore, the Gorenstein projective modules share many nice properties of the classical projective module (see, for instance, [4, 8, 7]). In this paper, we show that some of these results remain true whenever we consider, in Definition 1.1, \( Q \) to be in any class of modules containing all projective modules. Namely, we define \( \mathcal{X} \)-Gorenstein projective modules, where \( \mathcal{X} \) is a class of \( R \)-modules that contains all projective \( R \)-modules (see Definition 2.1). In Proposition 2.2 we characterize the \( \mathcal{X} \)-Gorenstein projective modules. Our main result is Theorem 2.3, in which, we study the behavior of the notion of \( \mathcal{X} \)-Gorenstein projective modules in short exact sequences. We end the paper with a characterization of rings over which every \( R \)-module is \( \mathcal{X} \)-Gorenstein projective. These rings are particular cases of the well-known quasi-Frobenius rings.

## 2 \( \mathcal{X} \)-Gorenstein projective modules

In this paper we investigate the following generalization of Gorenstein projective modules.

**Definition 2.1** Let \( \mathcal{X} \) be a class of \( R \)-modules that contains all projective \( R \)-modules. An \( R \)-module \( M \) is called \( \mathcal{X} \)-Gorenstein projective, if there exists
an exact sequence of projective $R$-modules $P = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ such that $M \cong \text{Im}(P_0 \to P^0)$ and $\text{Hom}_R(P, F)$ is exact whenever $F \in \mathcal{X}$.

The sequence $P$ is called an $\mathcal{X}$-complete projective resolution.

We start with the following characterization of an $\mathcal{X}$-Gorenstein projective module.

**Proposition 2.2** For an $R$-module $M$, the following conditions are equivalent:

1. $M$ is $\mathcal{X}$-Gorenstein projective.
2. i) $\text{Ext}^i_R(M, F) = 0$ for every $F \in \mathcal{X}$ and every $i > 0$;
   
   ii) There exists an exact sequence of $R$-modules $Q = 0 \to M \to P_0 \to P_1 \to \cdots$, where each $P_i$ is projective, such that $\text{Hom}_R(Q, F)$ is exact for every $F \in \mathcal{X}$.

3. There exists a short exact sequence of $R$-modules $0 \to M \to P \to N \to 0$, where $P$ is projective and $N$ is $\mathcal{X}$-Gorenstein projective.

4. There exists a family of short exact sequences of $R$-modules $0 \to M_i \to P_i \to M_{i+1} \to 0$ $(i \in \mathbb{Z})$, where each $P_i$ is projective and $M_0 = M$, such that $\text{Ext}^1_R(M_i, F) = 0$ for every $F \in \mathcal{X}$ and every $i \in \mathbb{Z}$.

**Proof.** The proof of the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) is analogous to the ones of the Gorenstein projective counterpart (see [4, 8]).

The implication (3) $\Rightarrow$ (4) is obvious.

To end, we prove the implication (3) $\Rightarrow$ (2). Let $F \in \mathcal{X}$. Applying the functor $\text{Hom}_R(-, F)$ to the exact sequence $0 \to M \to P \to N \to 0$, we get the long exact sequence: $\cdots \to \text{Ext}^i_R(N, F) \to \text{Ext}^i_R(P, F) \to \text{Ext}^i_R(M, F) \to \cdots$. For every $i > 0$, we have: $\text{Ext}^i_R(N, F) = 0$ (since $N$ is $\mathcal{X}$-Gorenstein projective and by the equivalent (1) $\Leftrightarrow$ (2)). Also, we have $\text{Ext}^i_R(P, F) = 0$ (since $P$ is projective). Then, $\text{Ext}^i_R(M, F) = 0$ for every $i > 0$.

It remains to prove (ii). Since $N$ is $\mathcal{X}$-Gorenstein projective and by the equivalent (1) $\Leftrightarrow$ (2), there exists an exact sequence of $R$-modules $P = 0 \to N \to P_0 \to P_1 \to \cdots$, where each $P_i$ is projective, such that $\text{Hom}_R(P, F)$ is exact for all $R$-modules $F \in \mathcal{X}$. Assembling this sequence with the short exact sequence $0 \to M \to P \to N \to 0$ we get the following exact sequence $Q = 0 \to M \to P \to P_0 \to P_1 \to \cdots$ such that the sequence $\text{Hom}_R(Q, F)$ is exact for every $R$-module $F \in \mathcal{X}$, as desired. \qed

The following result, which investigates the behavior of $\mathcal{X}$-Gorenstein projective modules in short exact sequences, generalizes [8, Theorem 2.5].
**Theorem 2.3**

1. Let \( 0 \to A \to B \to C \to 0 \) be a short exact sequences of \( R \)-modules, where \( C \) is \( \mathcal{X} \)-Gorenstein projective. Then, \( A \) is \( \mathcal{X} \)-Gorenstein projective if and only if \( B \) is \( \mathcal{X} \)-Gorenstein projective.

2. Let \((M_i)_{i \in I}\) be a family of \( R \)-modules. Then, \( \bigoplus_{i \in I} M_i \) is \( \mathcal{X} \)-Gorenstein projective if and only if \( M_i \) is \( \mathcal{X} \)-Gorenstein projective for every \( i \in I \).

**Proof.** The equivalences of both (1) and (2) can be proved similarly to the one of [8, Theorem 2.5]. Here, we give a new and simple proof of the “only if” part of (1). Then, assume that \( B \) is \( \mathcal{X} \)-Gorenstein projective. By Proposition 2.2 (1) \( \iff \) (3), there exists an exact sequence of \( R \)-modules \( 0 \to B \to P \to G \to 0 \), where \( P \) is projective and \( G \) is \( \mathcal{X} \)-Gorenstein projective. Consider the following pushout diagram:

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\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & A & B & C & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & A & P & C' & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
G & \underarc{\rightarrow} & G \\
0 & 0 & 0 & 0 & 0
\end{array}
\]
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Applying the “if” part to the right vertical short exact sequence, we get that \( C' \) is \( \mathcal{X} \)-Gorenstein projective. Therefore, use the equivalence (1) \( \iff \) (3) of Proposition 2.2 and the middle horizontal short exact sequence to get that \( A \) is \( \mathcal{X} \)-Gorenstein projective.

We end the paper with a characterization of rings over which every \( R \)-module is \( \mathcal{X} \)-Gorenstein projective. These rings are particular cases of the well-known quasi-Frobenius rings.

**Proposition 2.4** Every \( R \)-module is \( \mathcal{X} \)-Gorenstein projective if and only if every \( R \)-module in \( \mathcal{X} \) is injective.

In particular, if the above equivalence conditions are satisfied, then \( R \) is quasi-Frobenius.

**Proof.** First, from [3, Theorem 2.2] and its proof, if one of the equivalence conditions are satisfied, then \( R \) is quasi-Frobenius.
Now, assume that every $R$-module is $\mathcal{X}$-Gorenstein projective. Then, from Proposition 2.2, $\text{Ext}_R^i(M, F) = 0$ for every $R$-module $M$, every $F \in \mathcal{X}$, and every $i > 0$. Then, every $F$ in $\mathcal{X}$ is injective.

Conversely, consider an $R$-module $M$. Let $\cdots \to P_1 \to P_0 \to M \to 0$ and $0 \to M \to I_0 \to I_1 \to \cdots$ be projective and injective resolutions of $M$. Since, by the reason above, $R$ is quasi-Frobenius, every injective $R$-module is projective. Then, the above injective resolution is a right projective resolution of $M$. Now, assembling the two above resolutions, we get the following exact sequence: $\cdots \to P_1 \to P_0 \to I_0 \to I_1 \to \cdots$. Since, by hypothesis, every $R$-module in $\mathcal{X}$ is injective, the above exact sequence is clearly an $\mathcal{X}$-complete projective resolution, as desired. \[\square\]

References


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