On 3-Bézout Rings

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Abstract

In this paper, we study the notion of “3-Bézout ring”. We establish the transfer of this notion to homomorphic image and trivial ring extensions and provide a class of 3-Bézout rings which are not 2-Bézout rings.

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1 Introduction

All rings considered below are commutative with unit and all modules are unital. Let $R$ be a commutative ring, and let $M$ be an $R$-module. For any positive integer $n$, we say that $M$ is $n$-presented whenever there is an exact sequence:

$$F_n \rightarrow F_{n-1} \rightarrow \ldots \rightarrow F_0 \rightarrow M \rightarrow 0$$

of $R$-modules in which each $F_i$ is a finitely generated free $R$-module. In particular, 0-presented and 1-presented $R$-modules are respectively finitely generated and finitely presented $R$-modules. See for instance [4].

Recall that a ring $R$ is called Bézout if every finitely generated ideal $I$ of $R$ is principal. For a positive integer $n > 0$, we introduce a new concept of a “$n$-Bézout” ring. A ring $R$ is called $n$-Bézout if every $(n-1)$-presented ideal of $R$ is principal. 1-Bézout rings are exactly Bézout rings. Clearly, an $(n-1)$ Bézout ring is also an $n$-Bézout ring.

Let $A$ be a ring, $E$ be an $A$-module and $R := A \times E$ be the set of pairs $(a, e)$ with pairwise addition and multiplication given by: $(a, e)(b, f) = (ab, af + be)$. $R$ is called the trivial ring extension of $A$ by $E$. Considerable work, part of it summarized in Glaz’s book [4] and Huckaba’s book [5], has been concerned
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with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory. See for instance [1, 4, 5, 6].

In this paper, we establish the transfer of the ”3-Bézout” notion to homomorphic image and trivial ring extensions and provide a class of 3-Bézout rings which are not 2-Bézout rings.

2 Main Results

First, we study the homomorphic image of 3-Bézout rings which is a first main result in this paper.

**Theorem 2.1** Let \( R \) be a 3-Bézout ring and \( I \) be a finitely presented ideal of \( R \). Then \( R/I \) is a 3-Bézout ring.

To prove this Theorem we need the following Lemma.

**Lemma 2.2** Let \( R \) be a ring, \( I \) be a finitely generated ideal of \( R \), and let \( E \) be a finitely presented \( R/I \)-module. Then \( E \) is a finitely presented \( R \)-module.

**Proof.** Let \( E \) be a finitely presented \( R/I \)-module. Consider the exact sequence of \( R/I \)-modules:

\[
0 \rightarrow K \rightarrow F_0 \rightarrow E \rightarrow 0\quad (\ast)
\]

where \( F_0 \) is a finitely generated free \( R/I \)-module and \( K \) is a finitely generated \( R/I \)-module. Hence, \( K \) is a finitely generated \( R \)-module since \( K \) is a finitely generated \( R/I \)-module and \( R/I \) is a finitely generated \( R \)-module. On the other hand, \( F_0 \) is a finitely presented \( R \)-module since \( F_0 \) is a finitely generated free \( R/I \)-module and \( R/I \) is a finitely presented \( R \)-module (by the exact sequence of \( R \)-modules \( 0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0 \) and since \( I \) is a finitely generated ideal of \( R \)). Therefore, \( E \) is a finitely presented \( R \)-module by the exact sequence \( (\ast) \).

**Proof of Theorem 2.1.** Let \( J/I \) be a 2-presented ideal of \( R/I \), where \( I \subseteq J \) are ideals of \( R \). We claim that \( J/I \) is a 2-presented \( R \)-module. Indeed, consider the exact sequence of \( R/I \)-modules:

\[
0 \rightarrow E \rightarrow F_0 \rightarrow J/I \rightarrow 0\quad (\ast)
\]
where $F_0$ is a finitely generated free $R/I$-module and $E$ is a finitely presented $R/I$-module. Hence, $E$ is a finitely presented $R$-module by Lemma 2.2. On the other hand, $F_0$ is a 2-presented $R$-module since $F_0$ is a finitely generated free $R/I$-module and $R/I$ is a 2-presented $R$-module (by the exact sequence of $R$-modules $0 	o I 	o R 	o R/I 	o 0$ and since $I$ is a finitely presented ideal of $R$). Therefore, $J/I$ is a 2-presented $R$-module by the exact sequence $(\ast)$. Hence, $E$ is a finitely presented $R/I$-module by Lemma 2.2. On the other hand, $F_0$ is a 2-presented $R$-module since $F_0$ is a finitely generated free $R/I$-module and $R/I$ is a 2-presented $R$-module (by the exact sequence $0 	o I 	o R 	o R/I 	o 0$ and since $I$ is a finitely presented ideal of $R$). Therefore, $J/I$ is a 2-presented $R$-module by the exact sequence $(\ast)$. Now, the exact sequence of $R$-modules:

\[ 0 \to I \to J \to J/I \to 0 \]

shows that $J$ is a 2-presented ideal of $R$ since $I$ is a finitely presented ideal of $R$ and $J/I$ is a 2-presented $R$-module. Hence, $J$ is a principal ideal of $R$ since $R$ is a 3-Bézout ring; let $J := Ra$ for some $a \in J$. Therefore, $J/I = (R/I)(a + I)$ is a principal ideal of $R/I$ and this completes the proof of Theorem 2.1.

**Corollary 2.3** Let $R$ be a ring and $R[X]$ be the polynomial ring over $R$. If $R[X]$ is a 3-Bézout ring, then so is $R$.

**Proof** Clear by Theorem 2.1 since $R := R[X]/(X)$ and $(X)$ is infinitely presented ideal of $R[X]$.

The condition "$I$ is a finitely presented ideal of $R$" is necessary in Theorem 2.1 (see Example 2.6).

Now, we study a particular trivial ring extension and provide a class of 3-Bézout rings which are not 2-Bézout rings; and give then the second main result in this paper.

**Theorem 2.4** Let $(A, M)$ be a local ring and let $R := A \propto (A/M)$ be the trivial ring extension of $A$ by $A/M$. Then:
1) $R$ is a 3-Bézout ring provided $M$ is not a finitely generated ideal of $A$.
2) $R$ is not a 2-Bézout ring provided $A$ is not a 2-Bézout ring.

**Proof.** 1) Assume that $M$ is not a finitely generated ideal of $A$. We claim that there exists no proper 2-presented ideal of $R$ and this suffices to show that $R$ is a 3-Bézout ring. Deny. Let $J$ be a 2-presented proper ideal of $R$. Hence, $R/J$ is a 3-presented $R$-module by the exact sequence of $R$-modules $0 \to J \to R \to R/J \to 0$. Therefore, $R/J$ is a projective $R$-module by [6, Theorem 1.1(1)] and so $R/J$ is a free $R$-module since $R$ is local, a contradiction since $J(R/J) = 0$. Hence, there is no proper 2-presented ideal of $R$ and so $R$
is a 3-Bézout ring.

2) Assume that $A$ is not a 2-Bézout ring and let $I := \sum_{i=1}^{n} Ab_i$ be a finitely presented proper ideal of $A$ which is not a principal ideal. Set $J := \sum_{i=1}^{n} R(b_i, 0)(= I \propto 0)$ and consider the exact sequence of $R$-modules:

$$0 \to \text{Ker}(u) \to R^n \xrightarrow{u} J \to 0(*)$$

where $u((a_i, e_i)_{i=1,...,n}) = \sum_{i=1}^{n} (a_i, e_i)(b_i, 0) = (\sum_{i=1}^{n} a_i b_i, 0)$. Hence, $\text{Ker}(u) = V \propto (A/M)^n$, where $V = \{(a_i)_{i=1,...,n} \in A^n/\sum_{i=1}^{n} a_i b_i = 0\}$. Clearly, $V$ is a finitely generated $A$-module since $I$ is a finitely presented ideal and by the exact sequence of $R$-modules:

$$0 \to V \to A^n \xrightarrow{v} I \to 0(*)$$

where $v((a_i)_{i=1,...,n}) = \sum_{i=1}^{n} a_i b_i$. Therefore, $\text{Ker}(u)(= V \propto (A/M)^n)$ is a finitely generated $R$-module and so $J$ is a finitely presented ideal of $R$ by the exact sequence $(*)$. But $J$ is not principal ideal of $R$ since $I$ is not a principal ideal of $A$ and $J = I \propto 0$. Hence, $R$ is not a 2-Bézout ring and this completes the proof of Theorem 2.4.

Now, we are able to construct the first class of non-2-Bézout rings which are 3-Bézout rings.

Example 2.5 Let $A = K[[X_1,\ldots,X_n,\ldots]] = K + M$ be the formal power series, where $K$ is a field, $(X_i)_{i=1,\ldots,\infty}$ are indeterminates over $K$ and $M = \sum_i AX_i$ and let $R := A \propto (A/M)$ be the trivial ring extension of $A$ by $A/M$. Then:

1) $R$ is a 3-Bézout ring.
2) $R$ is not a 2-Bézout ring.

Proof. 1) By Theorem 2.4(1) since the maximal ideal $M = \sum_i AX_i$ of $A$ is not finitely generated.

2) By Theorem 2.4(2) since for example $AX_1 + AX_2$ is an infinitely presented ideal of $A$ (since $A$ is a coherent ring) but it is not a principal ideal of $A$.

The condition ”$I$ is a finitely presented ideal of $R$” is necessary in Theorem 2.1 as the following example shows.
Example 2.6  Let \((A, M)\) and \(R\) as in Example 2.5 and let \(I := 0 \otimes (A/M) = R(0,1)\). Then:

1) \(R\) is a 3-Bézout ring by Example 2.5(1).

2) \(I\) is a finitely generated ideal of \(R\) which is not finitely presented since \(M\) is not finitely generated and by the exact sequence of \(R\)-modules:

\[0 \to M \otimes (A/M) \to R \xrightarrow{u} I := R(0,1) \to 0.\]

3) \(R/I(= A)\) is not 3-Bézout ring since \(AX_1 + AX_2\) is an infinitely presented ideal of \(A\) (since \(A\) is coherent) which is not a principal ideal of \(A\).

References


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