On $f$-Best Approximation in Quotient Topological Vector Spaces

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Abstract

Assume $f$ be a real valued function on a topological vector space $X$. We extend known notions of $f$-best approximation in quotient this spaces. Sufficient conditions for the existence and uniqueness of $f$-best approximation are obtained.

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1 Introduction

The notion of $f$-best approximation in a vector space $X$ was given by Breckner and Brosowski [1] and in a Hausdorff topological space $X$ by Narang [4], [5]. Taking $X$ be a Hausdorff locality convex topological vector space and $f$ to be a continues sublinear functional on $X$, certain results on best approximation relative to the functional $f$ were proved in [1], [2]. Here we shall also discuss the existence, uniqueness and topological properties of $f$-best approximation sets in quotient spaces.

2 Preliminaries

Let $X$ be a Hausdorff topological vector space over $\mathbb{R}$ and $f$ a real valued function. Let $K$ be a nonempty closed subset of $X$ and $x \in X$.

Element $k_0 \in K$ is said to be an $f$-best approximation to $x$ in $K$ if

$$f(x - k_0) = f_K(x) = \inf \{f(x - k) : k \in K\}.$$
We denote by $P^f_k(x)$ the collection of all such $k_0 \in K$. The set $K$ is said to be $f$-proximinal if $P^f_k(x)$ is nonempty for each $x \in X$, $f$-semi-Chebyshev if $P^f_k(x)$ is at most singleton for any $x \in X$ and $f$-Chebyshev if $P^f_k(x)$ is exactly singleton for each $x \in X$.

**Remark 2.1** When $X$ is a normed linear space over the field of real numbers and $f(x, y) = \|x - y\|$ for all $x, y \in X$, the notions introduced above coincide with the corresponding notions that already exists in literature.

**Example 2.2** Let $X = \mathbb{R}^2$ and $K = \{(x_1, x_2) \in \mathbb{R}^2 | \exp x_1 + \exp x_2 \leq 20\}$ and let $f(x, y) = \exp(-x_1 - x_2)$. It is easily verified that $P^f_k(0, 0) = \{\ln 10, \ln 10\}$.

**Definition 2.3** $f$ is said to be inf-compact (resp. inf-locally compact) if the sub-level sets $S_\lambda = \{x \in X : f(x) \leq \lambda\}$ are compact (resp. locally compact) for each $\lambda \in \mathbb{R}$. $f$ is said to be inf-bounded if the sub-level sets $S_\lambda$ are bounded for each $\lambda \in \mathbb{R}$. $K$ is said to be $f$-boundedly compact if $(x - K) \cap S_\lambda$ is compact for each $x \in X$ and $\lambda \in \mathbb{R}$. $K$ is said to be $\gamma$-compact if for each $x \in X$, there exists $\gamma \in \mathbb{R}$ such that $\gamma > f_K(x)$ and $(x - K) \cap S_\gamma$ is compact.

**Definition 2.4** The set $K$ is said to be $f$-inf-compact if for each $x \in X$, each minimizing net $k_\alpha$ in $K$ (i.e. $f(x - k_\alpha) \to f_K(x)$) has a $f$-convergent subnet in $K$ (i.e. there exists subnet $\{k_{\alpha_\beta}\}$ of $\{k_\alpha\}$ and $k_0 \in K$ such that $\lim_\beta f(k_{\alpha_\beta} - k_0) = 0$).

**Definition 2.5** A subset $K$ of $X$ is called $f$-compact if every net in $K$ has a $f$-convergent subnet in $K$.

**Definition 2.6** A linear subspace $K$ of $X$ is called $f$-quasi-Chebyshev if $P^f_k(x)$ is nonempty and $f$-compact set in $X$ for every $x \in X$.

**Proposition 2.7** Consider the following statements:

1. $f$ is inf-compact;
2. $K$ is $f$-boundedly compact;
3. $K$ is $\gamma$-compact;
4. $K$ is $f$-inf-compact;
5. $K$ is $f$-proximinal.

We have $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$.

**Proof.** See [2].

**Definition 2.8** A function $f : X \to \mathbb{R}$ is homogeneous if $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{R}$ and $x \in X$. $f$ is called sublinear if $f(x + y) \leq f(x) + f(y)$ for all $x, y \in X$. $f$ is called ultra function if $f(x + y) \leq \max\{f(x), f(y)\}$ for all $x, y \in X$. 
3 Set of $f$-Approximations

**Theorem 3.1** Let $X$ be a topological vector space and $f$ be a real valued function.

a) If $K$ is a subset of $X$, then
1) $f_{K+y}(x+y) = f_K(x)$ for all $x,y \in X$.
2) $P^f_{K+y}(x+y) = P^f_K(x) + y$ for all $x,y \in X$.
3) $K$ is $f$-proximinal (f-chebyshev) if and only if $K+y$ is $f$-proximinal (f-chebyshev) for every $y \in X$.

Furthermore if $f$ is homogeneous, then
4) $f_{\alpha K}(\alpha x) = \alpha f_K(x)$ for all $x \in X$ and $\alpha \geq 0$.
5) $P^f_{\alpha K}(\alpha x) = \alpha P^f_K(x)$ for all $x \in X$ and $\alpha \geq 0$.
6) $K$ is $f$-proximinal (f-chebyshev) if and only if $\alpha K$ is $f$-proximinal (f-chebyshev) for all $\alpha \geq 0$.

b) If $M$ is a subspace of $X$, then
1) $f_M(x+y) = f_M(x) + y$ for all $x,y \in X$.
2) $P^f_M(x+y) = P^f_M(x) + y$ for all $x,y \in X$.

Furthermore if $f$ is homogeneous, then
3) $f_M(\alpha x) = \alpha f_M(x)$ for all $x \in X$ and $\alpha \in \mathbb{R}$.
4) $P^f_M(\alpha x) = \alpha P^f_M(x)$ for all $x \in X$ and $\alpha \in \mathbb{R}$.

Furthermore if $f$ is sublinear and positive, then
5) $f_M(x+y) \leq f_M(x) + f_M(y)$

**Proof.** a) (1) Let $x,y \in X$, then

$$f_{K+y}(x+y) = \inf_{k \in K} f(x+y-(k+y)) = f_K(x)$$

(2) $k_0 + y \in P^f_{K+y}(x+y)$ if and only if $f_{K+y}(x+y) = f(x+y-(k_0+y))$ and by (1) if and only if $f_K(x) = f(x-k_0)$. It is true if and only if $k_0 \in P^f_K(x)$. Thus $P^f_{K+y}(x+y) = P^f_K(x) + y$.

(3) It is clear by (2).

(4) $f_{\alpha K}(\alpha x) = \inf_{k \in K} f(\alpha x - \alpha k) = \alpha \inf_{k \in K} f(x-k) = \alpha f_K(x)$

(5) If $\alpha = 0$, the result is true. Thus assume that $\alpha > 0$. $k_0 \in P^f_{\alpha K}(\alpha x)$ if and only if $k_0 \in \alpha K$ and $f(\alpha x - k_0) = f_{\alpha K}(\alpha x)$ if and only if $\frac{1}{\alpha}k_0 \in K$ and $\alpha f(x - \frac{1}{\alpha}k_0) = f_{\alpha K}(x)$ and this implies that $\frac{1}{\alpha}k_0 \in P^f_K(x)$. So $k_0 \in \alpha P^f_K(x)$.

b) (1),(2),(3) and (4) is immediate consequence of (1),(2),(3) and (4) part (a) and this fact that $M+y=M$ and $\alpha M = M$ for all $y \in M$ and $\alpha \neq 0$.

(5)

$$f_M(x+y) = \inf_{m \in M} f(x+y-m) = \inf_{m,\hat{m} \in M} f(x+y-(m+\hat{m}))$$

$$\leq \inf_{m,\hat{m} \in M} (f(x-m)+f(y-\hat{m}))$$

$$= \inf_{m \in M} f(x-m) + \inf_{\hat{m} \in M} f(y-\hat{m}) = f_M(x) + f_M(y)$$
For a subset $K$ of $X$, let $\hat{K}_f = \{x \in X | f_K(x) = f(x)\}$. Now we have the following characterizations of a $f$-proximinal subspace $K$ of $X$.

**Theorem 3.2** If $K$ is a subspace of $X$ and $f$ is a real function, then $K$ is $f$-proximinal if and only if $X = K + \hat{K}_f$.

**Proof.** If $K$ is proximinal and $x \in X$, then there exists $k_0 \in K$ such that $f_K(x) = f(x - k_0)$. Hence $x - k_0 \in \hat{K}_f$. Thus $x = k_0 + (x - k_0) \in K + \hat{K}_f$. Therefore $X = K + \hat{K}_f$. For converse let $x \in X$. Then there exists $k_1 \in K$ and $k_2 \in \hat{K}_f$ such that $x = k_1 + k_2$. Thus $x - k_1 = k_2 \in \hat{K}_f$. Hence $f_K(x - k_1) = f(x - k_1)$. Since $K$ is subspace $f(x - k_1) = f_K(x)$. Thus $k_1 \in P^f_K(x)$ and $K$ is $f$-proximinal.

**Definition 3.3** A subset $K$ of a topological space $X$ is called $f$-closed if for all sequence $\{k_n\}$ of $K$ and $x \in X$ such that $f(x - k_n) \longrightarrow 0$, we have $x \in K$.

**Theorem 3.4** Let $f$ be a positive function and $x = 0 \iff f(x) = 0$. Then $K$ is $f$-proximinal, if and only if $K$ is $f$-closed and $\pi(\hat{K}_f) = \frac{X}{K}$.

**Proof.** Since $f$ is symmetric and ultra function, then $f(0) \leq \max\{f(x), f(-x)\}$, for any $x \in X$. Thus $f(x) \geq 0$ for all $x \in X$. Now let $K$ is $f$-proximinal and $\{k_n\} \subset K, x \in X$ and $f(x - k_n) \longrightarrow 0$. Then $f_K(x) = \inf_{k \in K} f(x - k) = 0$. Since $K$ is $f$-proximinal then there exists $k_0 \in K$ such that $f_K(x) = f(x - k_0)$. Hence $f(x - k_0) = 0$. Thus $x = k_0 \in K$. Therefore $K$ is $f$-closed. Now let $x \in X$ and $k_0 \in P^f_K(x)$. Then $x - k_0 \in \hat{K}_f$. Therefore $x + K = x - k_0 + K \in \frac{K}{K} = \frac{\hat{K}_f}{K}$. Hence $\pi(\hat{K}_f) = \frac{X}{K}$. Conversely let $\pi(\hat{K}_f) = \frac{X}{K}$. Thus if $x \in X$, then $x + K = y + K$ for some $y \in \hat{K}_f$. Hence $x - y = k_0$ for some $k_0 \in K$. Thus $x = y + k_0 \in \hat{K}_f + K$, Therefore $X = \hat{K}_f + K$. Hence $K$ is $f$-proximinal.

**Definition 3.5** A subset $A$ of $X$ is called $f$-bounded if $f(a) \leq M$ for all $a \in A$, for some $M > 0$.

Let $K$ be a $f$-proximinal subspace of $X$. Define $\varphi_K : \frac{X}{K} \longrightarrow 2^{\hat{K}_f}$ by

$$\varphi_K(x + K) = x - P^f_K(x).$$

$\varphi_K$ is well-defined because of for each $y \in P^f_K(x)$ we have $x - y \in \hat{K}_f$ and if $x_1 + K = x_2 + K$, then $x_1 - x_2 \in K$ and therefore $P^f_K(x_1) = P^f_K(x_2) + k$ where $x_1 - x_2 = k$. Thus $\varphi_K(x_1 + K) = \varphi_K(x_2 + K)$.

Also since $K$ is $f$-proximinal, then for all $x + K \in \frac{X}{K}$, we have $\varphi_K(x + K) \neq \phi$. 
Let \( k_1, k_2 \in P^f_K(x) \), then \( f_K(x) = f_K(x - k_1) = f_K(x - k_2) \). So if \( f \) is homogenous and ultra function, then

\[
f(x - (\lambda k_1 + (1 - \lambda)k_2)) = f(\lambda(x - k_1) + (1 - \lambda)(x - k_2)) \\
\leq \max\{f(\lambda(x - k_1)), f((1 - \lambda)(x - k_2))\} \\
= \max\{\lambda f(x - k_1), (1 - \lambda)f(x - k_2)\} \\
\leq f(x - k_1) = f(x - k_2) = f_K(x) \\
\implies f(x - (\lambda k_1 + (1 - \lambda)k_2)) = f_K(x)
\]

Thus \( P^f_K(x) \) is convex. Also if \( f \) is sublinear and symmetric, \( x \in X \) and \( k_0 \in P^f_K(x) \), then

\[
f(k_0) = f(k_0 - x + x) \leq f(x - k_0) + f(x) = f_K(x) + f(x).
\]

So \( P^f_K(x) \) is \( f \)-bounded.

Therefore we get:

**Lemma 3.6**

1) If \( K \) is \( f \)-proximinal subspace of \( X \), then \( \varphi_K(x + K) \neq \phi \) for all \( x \in X \).

2) If \( f \) is sublinear and symmetric, then \( \varphi_K(x + K) \) is \( f \)-bounded.

3) If \( f \) is homogenous and ultra function, then \( \varphi_K(x + K) \) is convex.

**Lemma 3.7** Let \( K \) be a \( f \)-proximinal subspace of \( X \) and \( y_i + M \in \hat{X}_M \) \( (i = 1, 2) \) and \( x_1 \in X \) be such that \( x_1 + K = y_1 + K \). Then there exists \( x_2 \in X \) such that \( x_2 + K = y_2 + K \) and \( f(x_1 - x_2) = f_K(y_1 - y_2) \).

**Proof.** Since \( x_1 + K = y_1 + K \), there exists \( k_1 \in K \) such that \( y_1 = x_1 - k_1 \). Since \( K \) is \( f \)-proximinal, there exists \( k_0 \in K \) such that \( f(y_1 - y_2 - k_0) = f_K(y_1 - y_2) \). Thus \( f(x_1 - k_1 - y_2 - k_0) = f_K(y_1 - y_2) \). Let \( x_2 = k_0 + k_1 + y_2 \). Then \( x_2 + K = y_2 + K \) and \( f(x_1 - x_2) = f_K(y_1 - y_2) \).

4 \( f \)-Approximation in Quotient Space

Let \( X \) be a topological vector space and \( K, M \) be subspace of \( X \) such that \( M \) is closed and \( M \subset K \). Let \( f : X \longrightarrow \mathbb{R} \) be a function. Define

\[
\tilde{f}(x + M) = \inf\{f(x + y)|y \in M\}.
\]

Then we have:
**Theorem 4.1**

1) If \( f \) is symmetric, then \( \tilde{f} \) is symmetric.

2) If \( f \) is sublinear and positive functional, then \( \tilde{f} \) is sublinear functional.

3) If \( f \) is ultra functional, then \( \tilde{f} \) is ultra functional.

4) If \( k_0 \) is an \( f \)-best approximation to \( x \) in \( K \), then \( k_0 + M \) is an \( \tilde{f} \)-best approximation to \( x + M \) in \( \frac{K}{M} \).

5) If \( k_0 + M \) is an \( \tilde{f} \)-best approximation to \( x + M \) from \( \frac{K}{M} \) and \( m_0 \) is a \( f \)-best approximation to \( x - k_0 \) from \( M \), then \( k_0 + m_0 \) is a \( f \)-best approximation to \( x \) from \( K \).

6) If \( M \) is \( f \)-proximinal in \( X \) and \( \frac{K}{M} \) is \( \tilde{f} \)-proximinal in \( \frac{X}{M} \), then \( K \) is \( f \)-proximinal in \( X \).

7) If \( M \) is an \( f \)-semi-chesbyshev in \( X \) and \( \frac{K}{M} \) is an \( \tilde{f} \)-semi-chesbyshev in \( \frac{X}{M} \), then \( K \) is \( f \)-semi-chesbyshev in \( X \).

8) If \( K \) is \( f \)-proximinal in \( X \), then \( \frac{K}{M} \) is \( \tilde{f} \)-proximinal in \( \frac{X}{M} \).

9) If \( M \) is \( f \)-chesbyshev in \( X \) and \( \frac{K}{M} \) is \( f \)-chesbyshev in \( \frac{X}{M} \), then \( K \) is \( f \)-chesbyshev in \( X \).

10) If \( M \) is \( f \)-proximinal in \( X \) and \( K \) is \( f \)-semi-chesbyshev in \( X \), then \( \frac{K}{M} \) is \( \tilde{f} \)-semi-chesbyshev in \( \frac{X}{M} \).

11) If \( M \) is \( f \)-proximinal in \( X \) and \( K \) is \( f \)-chesbyshev in \( X \), then \( \frac{K}{M} \) is \( \tilde{f} \)-chesbyshev in \( \frac{X}{M} \).

**Proof.**

1) Let \( x \in X \), then

\[
\tilde{f}(x + M) = \inf \{ f(x + y) \mid y \in M \} \\
= \inf \{ f(-x - y) \mid y \in M \} \\
= \inf \{ f(-x + z) \mid z \in M \} \\
= \tilde{f}(-x + M)
\]

2) If \( x_1, x_2 \in X \), we have

\[
\tilde{f}((x_1 + M) + (x_2 + M)) = \tilde{f}((x_1 + x_2) + M) \\
= \inf \{ f(x_1 + x_2 + y) \mid y \in M \} \\
= \inf \{ f(x_1 + y_1 + x_2 + y_2) \mid y_1, y_2 \in M \} \\
\leq \inf \{ f(x_1 + y_1) + f(x_2 + y_2) \mid y_1, y_2 \in M \} \\
\leq \inf \{ f(x_1 + y_1) \mid y_1 \in M \} + \inf \{ f(x_2 + y_2) \mid y_2 \in M \} \\
= \tilde{f}(x_1 + M) + \tilde{f}(x_2 + M)
\]

3) It is similar to (2).

4) If \( k_0 + M \) is not an \( \tilde{f} \)-best approximation to \( x + M \) in \( \frac{K}{M} \), then \( \tilde{f}(x - k_0 + M) \nless \tilde{f}(x - k + M) \), for all \( k \in K \). Hence there exists \( k_1 \in K \) such that \( \tilde{f}(x - k_1 + M) < \tilde{f}(x - k_0 + M) \). Since \( \tilde{f}(x - k_0 + M) \leq f(x - k_0) \), we have \( \tilde{f}(x - k_1 + M) < f(x - k_0) \). Thus for some \( m_0 \in M \), \( \tilde{f}(x - k_1 + m_0) < f(x - k_0) \).
Let $M, K$ be subspaces of a topological vector space $X$ and $M \subset K$. Also let $\pi : X \to \frac{X}{M}$ be the canonical map and $K$ be $f$-proximinal in $X$. Then

$$\pi(P^f_K(x)) \subset P^f_M(x + M).$$

Furthermore if $M$ is $f$-proximinal in $X$, then

$$\pi(P^f_K(x)) = P^f_M(x + M).$$

Proof. By theorem 3.1, part (4), it is clear that $\pi(P^f_K(x)) \subset P^f_M(x + M)$. Now, let $M$ be $f$-proximinal in $X$. By theorem 3.1, part (5), if $k_0 \in P^f_M(x + M)$
and $m_0 \in P_M^f(x - k_0)$, then $k_0 + m_0 \in P_M^f(x)$. Therefore

$$k_0 + M = k_0 + m_0 + M = \pi(k_0 + m_0) \in \pi(P_M^f(x))$$

Hence $P_M^f(x + M) \subseteq \pi(P_M^f(x))$.

**Theorem 4.3** Let $K$ and $M$ be subspaces of $X$ and $M \subset K$. Consider the following statements:

1. $f$ is inf-compact;
2. $K$ is $f$-boundedly compact;
3. $K$ is $\gamma$-compact;
4. $K$ is $f$-inf-compact;
5. $K$ is $f$-proximinal.
6. $\frac{K}{M}$ is $\tilde{f}$-proximinal.

Then we have $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$.

**Proof.** It is clear by proposition 1.7 and theorem 3.1.

## 5 $f$-quasi chebyshevily in quotient spaces

**Theorem 5.1** Let $f$ be a positive function, $M$ be a $f$-proximinal subspace of $X$ and $K$ be a $f$-quasi-chebyshev subspace of $X$ such that $M \subset K$. Then $\frac{K}{M}$ is $\tilde{f}$-quasi-chebyshev in $\frac{X}{M}$.

**Proof.** By theorem 3.1 part (4), $\frac{K}{M}$ is $\tilde{f}$-proximinal in $\frac{X}{M}$. Let $x \in X$ and $\{k_\alpha + M\}$ be a net in $P_M^f(x + M)$. Then by corollary 3.2 there exists $k'_\alpha \in P_K^f(x)\ such\ that\ k'_\alpha + M = k_\alpha + M$ for all $\alpha$. Since $K$ is $f$-quasi-chebyshev, then there exists a subnet $\{k'_{\alpha,\beta}\}$ of $\{k'_\alpha\}$ and $k_0 \in P_K^f(x)$ such that $f(k'_{\alpha,\beta} - k_0) \rightarrow 0$. Since $\tilde{f}(k'_{\alpha,\beta} - k_0 + M) \leq f(k'_{\alpha,\beta} - k_0)$ and $f$ is positive, then $\tilde{f}(k'_{\alpha,\beta} - k_0 + M) \rightarrow 0$. Hence $\tilde{f}((k'_{\alpha,\beta} + M) - (k_0 + M)) \rightarrow 0$. Thus $P_M^f(x)$ is $\tilde{f}$-compact and $\frac{K}{M}$ is $\tilde{f}$-quasi-chebyshev in $\frac{X}{M}$.

**Remark 5.2** Here there are some open problem such that as:

1) If $f$ is inf-compact. Is $\tilde{f}$, inf-compact?

2) If $K$ and $M$ are subspaces of $X$, $M \subset K$ and $K$ is $f$-boundedly compact, $\gamma$-compact, inf-compact, Is $\frac{K}{M}$ $\tilde{f}$-boundedly compact, $\gamma$-compact, inf-compact, respectively?
References


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