Results on Certain Subclasses of Analytic Functions

Related to Complex Order

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Abstract

In this paper we introduce a class $G (\lambda, \mu, A, B, b)$ of analytic functions $f (z)$ of complex order $b$, using convolution technique. We estimate the coefficient $a_n$ in our main result, this also generalize Ahuja [8]. Our next result establishes sufficient condition for the function $f (z)$ belonging to the class $G (\lambda, \mu, A, B, b)$ which generalize Chaudhary [1].

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1. Introduction

The analytic functions have played a very important role in the development of certain subclasses of analytic functions of a complex order. Many authors obtained some beautiful results regarding subclasses of analytic functions. To see this end we refer Chaudhary [1], Silvia [2], Silverman [3], Kim and Show [4], Shaqsi and Darus [5], Nasr and Aouf [6], Robertson [7], Ahuja [8], Chichra [9], Goel and Mehrok [10], Shukla and Dashrath [11] and Ruscheweyh [12].

Let \( f(z) \) be an analytic function in class \( A \) such that
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{where } f(z) \in A \text{ is analytic and univalent in the unit disk } U = \{z: |z| < 1\}.
\]

In this paper, we introduce a class \( G(\lambda, \mu, A, B, b) \) of analytic functions \( f(z) \) of complex order \( b \), by using convolution technique, as follows. A function \( f \) of \( A \) belongs to the class \( G(\lambda, \mu, A, B, b) \) if and only if there exists a function \( w \) belonging to the class \( H \) such that
\[
1 + \frac{1}{b} \left\{ \frac{z(D^\lambda f(z))}{D^\lambda f(z)} - 1 \right\} = (1 - \mu) + \mu \left\{ \frac{1 + Aw(z)}{1 + Bw(z)} \right\}, \quad z \in U
\]
(1.1)
where \(-1 \leq B < A \leq 1, 0 < \mu \leq 1, \lambda > -1,\)

2. Preliminaries

**Def. (2.1):** Let \( f(z) \) defined by \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and \( b \) is any non-zero complex number then \( D^\lambda f(z) \) defined by
\[
D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} \ast f(z) = \frac{z(z^{\lambda-1} f(z))^{(\lambda)}}{\lambda!}
\]
where \( \ast \) denotes the Hadamard product of two analytic functions.

**Def. (2.2):** If \( f(z) \) and \( g(z) \) are any two functions in class \( A \) such that
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n
\]
then the convolution or Hadamard product of \( f(z) \) and \( g(z) \) is denoted by \( f \ast g \), and is defined by the power series \( (f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \).

Now using the identity \( z(D^\lambda f(z))' = (\lambda + 1)D^{\lambda+1} f(z) - \lambda D^\lambda f(z) \) in (1.1),
we have
\[ 1 + \frac{\lambda + 1}{b} \left\{ \frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)} - 1 \right\} = (1 - \mu) + \mu \left\{ \frac{1 + Aw(z)}{1 + Bw(z)} \right\} , \quad z \in U \]  \hfill (2.3)

Clearly this can be seen that (1.1) and (2.3) are equivalent to
\[
\begin{align*}
\frac{z(D^\lambda f(z)' - 1)}{D^\lambda f(z)} < 1, & \quad z \in U \\
\mu(A - B)b - B \left\{ \frac{z(D^{\lambda+1}f(z)' - 1)}{D^{\lambda}f(z)} \right\} < 1, & \quad z \in U
\end{align*}
\]  \hfill (2.4)

and
\[
\begin{align*}
\frac{(\lambda + 1)}{\mu(A - B)b - (\lambda + 1)B} \left\{ \frac{(D^\lambda f(z) - 1)}{D^\lambda f(z)} \right\} < 1, & \quad z \in U \\
(\lambda + 1) \left\{ \frac{(D^\lambda f(z) - 1)}{D^\lambda f(z)} \right\} & \leq (\lambda + 1) \left\{ \frac{D^{\lambda+1}f(z) - 1}{D^{\lambda}f(z)} \right\},
\end{align*}
\]  \hfill (2.5)

respectively.

### 3. Main Results

Before giving our main result we quote a Lemma due to Robertson [7] and prove a Lemma which are needed in our investigation.

**Lemma (3.1) [7]:** Let \( h(z) = \sum_{p=q}^{\infty} \left| d_p \right|^2 z^p \) and \( H(z) = \sum_{p=q}^{\infty} \left| D_p \right|^2 z^p \), \( q \geq 0 \). If \( h(z) = w(z) H(z) \), where \( w(0) = 0 \) and \( \left| d w(z) \right| < 1 \) in \( U \), then \( dq = 0 \) and
\[
\sum_{p=q+1}^{k} \left| d_p \right|^2 \leq \sum_{p=q}^{k} \left| D_p \right|^2 , \quad (k = q+1, q+2...)
\]  \hfill (3.1.1)

**Lemma (3.2):** For a fixed integer \( k \), \( k \geq 3 \), let
\[
M_j = \frac{\mu(A - B) - (j - 2)B}{(\lambda + j - 1)^2} , \quad (j = 2, 3..., k)
\]
and
\[
c(\lambda, p) = \frac{(\lambda + 1)^{p-1}}{p-1!} , \quad (p = 2, 3, ...)
\]
\[ c(\lambda, p) = \frac{(\lambda + 1)(\lambda + 2)...(\lambda + p - 1)}{p-1!} , \quad (p = 2, 3, ...)
\]
then
\[
\frac{1}{\{(k-1)c(\lambda, k)\}^2} \left[ \frac{\mu^2 (A - B)^2 |b|^2}{(\lambda + 1)^3} + \sum_{p=2}^{k-1} \{\mu (A - B)b\} \right] \\
-(p-1) B^2 - (p-1)^2 \{c(\lambda, p)\}^2 \prod_{j=2}^{p} M_j = \prod_{j=2}^{k} M_j \]
\] (3.2.1)

**Proof:** We shall prove (3.2.1) by mathematical induction on \(k\). It can easily be seen that (3.2.1) holds for \(k = 3\). Now assume that (3.2.1) is valid for \(k = 4, 5, \ldots, t-1\), then for \(k = t\) the left side of (3.2.1) gives
\[
\frac{1}{\{(t-1)c(\lambda, t)\}^2} \left[ \frac{\mu^2 (A - B)^2 |b|^2}{(\lambda + 1)^3} + \sum_{p=2}^{t-1} \{\mu (A - B)b\} \right] \\
-(t-1) B^2 - (t-1)^2 \{c(\lambda, t)\}^2 \prod_{j=2}^{t-1} M_j = \prod_{j=2}^{t} M_j \]
\]
This concludes the proof of the above lemma.

**Coefficient estimate:**

**Theorem (3.3):** Let \(f(z)\) be an analytic function defined by
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]
belongs to the class \(G(\lambda, \mu, A, B, b)\). Then
\[
|a_2| \leq \frac{\mu(A - B)|b|}{(\lambda + 1)}, \quad (3.3.1)
\]
And if \(\mu(A - B)b - B \leq 1\) and \(n \geq 3\), then
\[
|a_n| \leq \frac{\mu(A - B)|b| (n-2)!}{(\lambda + 1)_{n-1}}, \quad (3.3.2)
\]
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Furthermore, if \( |\mu(A-B)b-(n-2)B| > (n-2), n \geq 3 \),

Let

\[
M = \left\lfloor \frac{\mu(A-B)b-(n-2)B}{n-2} \right\rfloor
\]

be the greatest integer less than or equal to the expression within the square bracket. Then

\[
|a_n| \leq \frac{1}{(\lambda+1)} \prod_{j=2}^{n} |\mu(A-B)b-(j-2)B| \quad \text{for } n = 3, 4, \ldots, M+2; \quad (3.3.3)
\]

and

\[
|a_n| \leq \frac{(n-2)!}{(M+1)!}(\lambda+1) \prod_{j=2}^{M+3} |\mu(A-B)b-(j-2)B| \quad \text{for } n > M+2 \quad (3.3.4)
\]

The bounds (3.3.1) and (3.3.3) are sharp for all admissible \( \lambda, \mu, A, B, b \) and for each \( n \).

**Proof:** From (1.1) we have

\[
z(D^j f(z)) - D^j f(z) = \left\{ \left\{ \mu(A-B)b + B \right\} D^j f(z) - Bz(D^j f(z)) \right\} w(z) \quad (3.3.5)
\]

Since \( D^j f(z) = z + \sum_{p=2}^{\infty} c(\lambda, p)a_p z^p \) it follows that (3.3.5) is equivalent to

\[
\sum_{p=2}^{\infty} (p-1)c(\lambda, p)a_p z^p = \left\{ \sum_{p=2}^{\infty} \left\{ \mu(A-B)b-(p-1)B \right\} c(\lambda, p)a_p z^p \right\} w(z)
\]

Where \( a_1 = 1 \). Now using Lemma (3.1) [8], we obtain

\[
\sum_{p=2}^{\infty} (p-1)^2 \left\{ c(\lambda, p) \right\}^2 |a_p|^2 \leq \sum_{p=1}^{\infty} \left\{ \mu(A-B)b-(p-1)B \right\} \left\{ c(\lambda, p) \right\}^2 |a_p|^2
\]

which simplifies to

\[
|a_n|^2 \leq \frac{1}{\{(n-1)c(\lambda, n)\}} \left[ \mu^2(A-B)^2 b^2 \right]
\]

\[+ \sum_{p=2}^{n-1} \left| \mu(A-B)b-(p-1)B \right|^2 - (p-1)^2 \left| c(\lambda, p) \right|^2 |a_p|^2 \]

(3.3.6)

for every \( n = 2, 3, \ldots, \).

For \( n = 2 \), we get

\[
|a_2|^2 \leq \left( \frac{\mu(A-B)b}{\lambda+1} \right)^2 \quad \text{which proves (3.3.1)}
\]

Now suppose that \( |\mu(A-B)b-B| \leq 1 \) and \( n \geq 3 \). Then it follows that

\[
|\mu(A-B)b-(n-2)B| \leq (n-2) \quad \text{and } n \geq 3.
\]

Since all the terms under the summation in (3.3.6) are non positive, we obtain

\[
|a_n| \leq \frac{\mu(A-B)b}{(n-1)c(\lambda, n)} , \quad n \geq 3 \quad \text{Which gives (3.3.2)}
\]
However, if \( |\mu(A-B)b-(n-2)B| > (n-2), \ n \geq 3 \), then all the terms under the summation in (3.3.6) are positive. We shall establish (3.3.3) for integer \( n \geq 3 \) and \( n \leq M+2 \), from (3.3.6) by mathematical induction. For \( n = 3 \), (3.3.6) contributes
\[
|a_3| \leq \frac{\mu(A-B)|b|\mu(A-B)b-B^2}{(\lambda+1)(\lambda+2)}, \text{ which proves (3.3.3) for } n = 3.
\]
Suppose (3.3.3) holds for \( n = 4, 5, \ldots, k-1 \). Then for \( n = k \), (3.3.6) yields
\[
|a_k|^2 \leq \frac{1}{(k-1)c(\lambda,n)} \left[ \mu^2 (A-B)^2 |b|^2 \right.
\]
\[
+ \sum_{p=2}^{k-1} \left\{ \left[ \mu(A-B)b-(p-1)B \right]^2 - (p-1)^2 \right\} \left\{ c(\lambda, p) \right\}^2 |a_p|^2 \bigg) \]
\[
\leq \frac{1}{(k-1)c(\lambda,n)} \left[ \mu^2 (A-B)^2 |b|^2 \right.
\]
\[
+ \sum_{p=2}^{k-1} \left\{ \left[ \mu(A-B)b-(p-1)B \right]^2 - (p-1)^2 \right\} \left\{ c(\lambda, p) \right\}^2 |a_p|^2 \bigg) \]
\[
+ \prod_{j=2}^{k} \left[ \frac{\mu(A-B)b-(j-2)B}{(\lambda+j-1)^2} \right]
\]
\[
= \sum_{j=2}^{k} \left| \frac{\mu(A-B)b-(j-2)B}{(\lambda+j-1)^2} \right|^2 \text{ by Lemma (3.2).}
\]
It is now easy to show that (3.3.3) holds for \( n \leq M+2 \). Finally suppose that \( n > M+2 \). Then we may write (3.3.6) as
\[
|a_n|^2 \leq \frac{1}{(n-1)c(\lambda,n)} \left[ \mu^2 (A-B)^2 |b|^2 \right.
\]
\[
+ \sum_{p=2}^{M+2} \left\{ \left[ \mu(A-B)b-(p-1)B \right]^2 - (p-1)^2 \right\} \left\{ c(\lambda, p) \right\}^2 |a_p|^2 \bigg) \]
\[
+ \sum_{p=M+3}^{n-1} \left\{ \left[ \mu(A-B)b-(p-1)B \right]^2 - (p-1)^2 \right\} \left\{ c(\lambda, p) \right\}^2 |a_p|^2 \bigg) \]
\[
\leq \frac{1}{(n-1)c(\lambda,n)} \left[ \mu^2 (A-B)^2 |b|^2 \right.
\]
\[
+ \sum_{p=2}^{M+2} \left\{ \left[ \mu(A-B)b-(p-1)B \right]^2 - (p-1)^2 \right\} \left\{ c(\lambda, p) \right\}^2 |a_p|^2 \bigg) \]
\[
+ \sum_{p=M+3}^{n-1} \left\{ \left[ \mu(A-B)b-(p-1)B \right]^2 - (p-1)^2 \right\} \left\{ c(\lambda, p) \right\}^2 |a_p|^2 .
\]
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\[
\prod_{j=2}^{p} \left(\frac{\mu(A - B)b - (j - 2)B}{(\lambda + j - 1)^2}\right)
\]

\[
= \left[\frac{(M + 2)c(\lambda, M + 3)}{(n - 1)c(\lambda, n)}\right]^2 \prod_{j=2}^{M+1} \left(\frac{\mu(A - B)b - (j - 2)B}{(\lambda + j - 1)^2}\right)
\]

\[
= \left[\frac{(n - 2)!}{(M + 1)!}(\lambda + 1)(\lambda + 2)...(\lambda + n - 1)\right]^2 \prod_{j=2}^{M} \left(\frac{\mu(A - B)b - (j - 2)B}{(\lambda + j - 1)^2}\right)
\]

by an application of Lemma (3.2) and (3.3.4) immediately follows from the above. The equality in (3.3.3) and (3.3.2) are attained for the function \( f_n(z) \) given by

\[
f_n(z) = \frac{z}{(1 - z)^{n+1}} = \begin{cases} z(1 + Bz)^{\mu(A - B)b / B(n - 1)}, & B \neq 0 \\ z \exp[\{\mu(A - B) + Bb\}], & B = 0 \end{cases}
\]

where \( |\mu(A - B)b - B| \leq 1 \).

Finally, the inequality (3.3.3) is sharp for the function \( f(z) \) given by

\[
f(z) = \frac{z}{(1 - z)^{n+1}} = \begin{cases} z(1 + Bz)^{\mu(A - B)b / B}, & B \neq 0 \\ z \exp[\{\mu(A - B) + Bb\}], & B = 0 \end{cases}
\]

where \( |\mu(A - B)b - (n - 2)B| > (n - 2), \quad n \geq 3 \).

Remark: [3.1] If we take \( \mu = 1 \), theorem (3.3) coincides with the corresponding result of Ahuja [8]

Sufficient condition:

Theorem (3.4): Let \( f(z) \) be an analytic function defined by \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) in \( U \). If

\[
\sum_{n=2}^{\infty} \left\{ (n - 1) + |\mu(A - B)b - (n - 1)B| \right\} c(\lambda, n)a_n \leq |\mu(A - B)b|
\]

holds for some \( \lambda \geq 0 \), then the function \( f(z) \) belongs to the class \( G(\lambda, \mu, A, B, b) \).

Proof: Suppose that the inequality (3.4.1) holds. Then, for \( z \in U \), we have

\[
|z(D^\lambda f(z)) - D^\lambda f(z) - [\mu(A - B)b + B] D^\lambda f(z) - Bz(D^\lambda f(z))|\
\]

\[
= \sum_{n=2}^{\infty} (n - 1)c(\lambda, n)a_n z^n - \mu(A - B)bz + \sum_{n=2}^{\infty} [\mu(A - B)b(n - 1)B] c(\lambda, n)a_n z^n
\]

\[
\leq \sum_{n=2}^{\infty} (n - 1)c(\lambda, n)a_n z^n - \mu(A - B)b[|z|^{n+1} - \sum_{n=2}^{\infty} [\mu(A - B)b(n - 1)B] c(\lambda, n)a_n z^n]
\]

\[
< \sum_{n=2}^{\infty} (n - 1)c(\lambda, n)a_n z^n - \mu(A - B)b + \sum_{n=2}^{\infty} [\mu(A - B)b - (n - 1)B] c(\lambda, n)a_n z^n
\]

\[
= \sum_{n=2}^{\infty} \left\{ (n - 1) + |\mu(A - B)b - (n - 1)B| \right\} c(\lambda, n)a_n - \mu(A - B)b
\]
$\leq 0$.

By the inequality (3.4.1). Hence the function $f(z)$ belongs to the class $G(\lambda, \mu, A, B, b)$.

**Remark (3.2):** If we take $\mu = 1, A = 1$ and $b = -1$, theorem (3.4.1) coincides with the corresponding result of Chaudhary [1].

**References**


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