A Generalization of Some Inequalities for the log-Convex Functions

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Abstract

In this paper, we obtain a generalization of a double inequality on the log-convex functions, obtained by E. Neuman [8]. Also, we rediscover some inequalities of gamma and \( q \)-gamma functions. Finally, we present some new inequalities of Riemann zeta function.

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1 Introduction

The collection of log convex functions features pretty interesting functions that are widely used in applied mathematics and in statistics. [4], [9]. They are often useful in finding bounds for special functions and their zeros. Many inequalities for special functions are statements about the monotonicity of certain quantities [3], [11], [12]. In this work, we will present a monotonicity property and some inequalities of the logarithmically convex functions. As special cases, we will get some inequalities involving the gamma function and the \( q \)-gamma function.

Recall [2] that if \( f(x) \) is a twice differentiable function then \( f(x) \) is convex iff \( f''(x) \geq 0 \) for all \( x \) of our interval and hence \( f'(x) \) is increasing function in
our interval. Also, a function $f(x)$ defined and positive on a certain interval is called log-convex if the function $\log f(x)$ is convex. The condition that $f(x)$ be positive is obviously necessary, for otherwise the function $\log f(x)$ could not be formed.

2 Main result.

**Theorem 2.1** Let $f(x)$ be a positive, differentiable and log-convex function defined on $[0, \infty)$. Let

$$g(x) = \frac{[f(cx + r)]^a}{[f(bx + s)]^k}, \quad a, c, r, k, b, s > 0$$

Then

1. $g(x)$ decreases on its domain if $c \leq b$, $r \leq s$ and $ac \leq kb$. Hence

$$\frac{[f(cy + r)]^a}{[f(by + s)]^k} \leq \frac{[f(cx + r)]^a}{[f(bx + s)]^k} \leq \frac{[f(r)]^a}{[f(s)]^k} \quad \forall \ 0 \leq x \leq y. \tag{2}$$

2. $g(x)$ increases on its domain if $c \geq b$, $r \geq s$ and $ac \geq kb$. Hence

$$\frac{[f(r)]^a}{[f(s)]^k} \leq \frac{[f(cx + r)]^a}{[f(bx + s)]^k} \leq \frac{[f(cy + r)]^a}{[f(by + s)]^k} \quad \forall \ 0 \leq x \leq y. \tag{3}$$

**Proof.** Logarithmic convexity of the function $f$ implies that its logarithmic derivative $\alpha(x) = \frac{f'(x)}{f(x)}$ is increasing function. Then

$$\alpha(x) \leq \alpha(y) \quad \forall \ 0 \leq x \leq y. \tag{4}$$

Also,

$$\frac{d}{dx} g(x) = g(x)[ac \alpha(cx + r) - kb \alpha(bx + s)]. \tag{5}$$

In case of $c \leq b$, $r \leq s$ and $ac \leq kb$, we get for $x \geq 0$ that

$$ac \alpha(cx + r) \leq kb \alpha(bx + s). \tag{6}$$

Then $g'(x) \leq 0$ and hence $g(x)$ is decreasing on its domain. In particular

$$g(y) \leq g(x) \leq g(0) \quad \forall \ 0 \leq x \leq y, \tag{7}$$

which is the inequality (2). The second part has similar proof.
3 Some particular cases.

3.1 Inequalities involving the gamma function

The Euler gamma function $\Gamma(x)$ is log-convex function for $x > 0$ and is defined by [2]

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > 0,$$

By using a geometrical method, C. Alsina and M. S. Tomás [1] studied an inequality involving the gamma function and proved the following double inequality

$$\frac{1}{n!} \leq \frac{[\Gamma(x + 1)]^n}{\Gamma(1 + nx)} \leq 1; \quad 0 \leq x \leq 1, \; n \in \mathbb{N}. \quad (8)$$

J. Sándor [10] extended this result to a more general case, and obtained the following inequality

$$\frac{1}{\Gamma(1 + a)} \leq \frac{[\Gamma(x + 1)]^a}{\Gamma(1 + ax)} \leq 1; \quad 0 \leq x \leq 1, \; a \geq 1 \quad (9)$$

by using the series representation of the digamma function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. L. Bougoffa [3] used the same method of Sándor [10] to prove that the function

$$f_1(x) = \frac{[\Gamma(cx + 1)]^a}{\Gamma(bx + 1)^k}; \quad x \geq 0, \; a \geq c > 0, \quad (10)$$

is decreasing, so the following double inequality satisfies

$$\frac{[\Gamma(c + 1)]^a}{\Gamma(b + 1)^k} \leq \frac{[\Gamma(cx + 1)]^a}{\Gamma(bx + 1)^k} \leq 1; \quad 0 \leq x \leq 1, \; a \geq c > 0. \quad (11)$$

Also, A. S. Shabani [11], used the same method of Sándor [10] to prove that the function

$$f_2(x) = \frac{[\Gamma(cx + r)]^a}{\Gamma(bx + s)^k}; \quad x \geq 0, \quad (12)$$

where $a, b, c, r, s, k$ are real numbers such that $0 < cx + r \leq bx + s, bk \geq ca > 0$ and $\psi(cx + r) > 0$ (or $\psi(bx + s) > 0$) is decreasing and for $0 \leq x \leq 1$ the following double inequality holds

$$\frac{[\Gamma(c + r)]^a}{\Gamma(b + s)^k} \leq \frac{[\Gamma(cx + r)]^a}{\Gamma(bx + s)^k} \leq \frac{[\Gamma(r)]^a}{\Gamma(s)^k}. \quad (13)$$

Also, in case $ca \geq bk > 0$ and $\psi(cx + r) < 0$ (or $\psi(bx + s) < 0$) the function $f_2(x)$ is decreasing and the inequality (13) holds.
**Remark 1:** Considering (2) with \( f(x) = \Gamma(x), \ a = b = n \in N, \ y = c = r = s = k = 1 \) we obtain inequality (8).

**Remark 2:** Considering (2) with \( f(x) = \Gamma(x), \ b = a, \ y = c = r = s = k = 1 \) we obtain inequality (9).

**Remark 3:** Considering (2) with \( f(x) = \Gamma(x), \ y = r = s = 1 \) we obtain the inequality (11) and the function \( f_1(x) \) will be decreasing.

**Remark 4:** Considering (2) with \( f(x) = \Gamma(x), \ y = 1 \) we obtain the inequality (13) and the function \( f_2(x) \) will be decreasing, where the conditions \( 0 < cx + r \leq bx + s \) and \( bk \geq ca > 0 \) will satisfy \( \forall x \geq 0 \) iff \( a, c, r, k, b, s > 0 \) also, the condition \( \psi(cx + r) > 0 \) or \( \psi(bx + s) > 0 \) will satisfies automatically because if \( f(x) = \Gamma(x) \) then \( \psi(x) = \alpha(x) \) which is increasing function.

**Lemma 3.1**
1. For all \( 0 \leq x \leq y, \ c \leq b, \ r \leq s, \ ac \leq kb \) and \( a, c, r, k, b, s > 0 \), we have
\[
\frac{[\Gamma(cx + r)]^a}{[\Gamma(by + s)]^k} \leq \frac{[\Gamma(cx + r)]^a}{[\Gamma(by + s)]^k} \leq \frac{[\Gamma(r)]^a}{[\Gamma(s)]^k}.
\]
(14)
2. For all \( 0 \leq x \geq y, \ c \geq b, \ r \geq s, \ ac \geq kb \) and \( a, c, r, k, b, s > 0 \), we have
\[
\frac{[\Gamma(r)]^a}{[\Gamma(s)]^k} \leq \frac{[\Gamma(cx + r)]^a}{[\Gamma(by + s)]^k} \leq \frac{[\Gamma(cx + r)]^a}{[\Gamma(by + s)]^k}.
\]
(15)

### 3.2 Inequalities involving the \( q \)-gamma function

The \( q \)-gamma function \( \Gamma_q(x) \) is log-convex function for \( x > 0 \) and is defined by [5]

\[
\Gamma_q(x) = \frac{(q, q)_\infty}{(q^x, q)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1,
\]

where the \( q \)-shifted factorials \( (a; q)_\infty = \prod_{i=0}^\infty (1 - aq^i) \). This function is a \( q \)-analogue of the gamma function since we have \( \lim_{q \to 1} \Gamma_q(x) = \Gamma(x) \).

T. Kim and C. Adiga [6] proved for \( 0 < q < 1 \)
\[
\frac{1}{\Gamma_q(1 + a)} \leq \frac{[\Gamma_q(x + 1)]^a}{\Gamma_q(1 + ax)} \leq 1; \quad 0 \leq x \leq 1, \ a \geq 1,
\]
(16)

Also, T. Mansour [7] proved for \( 0 < q < 1 \)
\[
\frac{[\Gamma_q(r)]^a}{[\Gamma_q(s)]^k} \leq \frac{[\Gamma_q(sx + r)]^a}{[\Gamma_q(rx + s)]^k} \leq [\Gamma_q(s + r)]^{a-k},
\]
(17)
where \( \psi_q(bx + s) > 0, \ sa \geq rk, \ 0 \leq x \leq 1, \ r \geq s > 0 \) and \( c, d \) are positive real numbers.

Also, A. S. Shabani [12], proved that the function
\[
f_3(x) = \frac{[\Gamma_q(cx + r)]^a}{[\Gamma_q(bx + s)]^k}; \quad x \geq 0,
\]
(18)
where \(a, b, c, r, s, k\) are real numbers such that \(0 < cx + r \leq bx + s, bk \geq ca > 0\) and \(\psi_q(cx + r) > 0\) (or \(\psi_q(bx + s) > 0\)) is decreasing and for \(0 \leq x \leq 1\) the following double inequality holds

\[
\frac{[\Gamma_q(c + r)]^a}{[\Gamma_q(b + s)]^k} \leq \frac{[\Gamma_q(cx + r)]^a}{[\Gamma_q(bx + s)]^k} \leq \frac{[\Gamma_q(r)]^a}{[\Gamma_q(s)]^k}. \tag{19}
\]

Also, in case \(ca \geq bk > 0\) and \(\psi_q(cx + r) < 0\) (or \(\psi_q(bx + s) < 0\)) the function \(f_3(x)\) is decreasing and inequality (19) holds.

**Remark 5:** Considering (2) with \(f(x) = \Gamma_q(x), b = a, y = c = r = s = k = 1\) we obtain inequality (16).

**Remark 6:** Considering (3) with \(f(x) = \Gamma_q(x), b = r, c = s, y = 1\) we obtain inequality (17).

**Remark 7:** Considering (2) with \(f(x) = \Gamma_q(x), y = 1\) we obtain inequality (19) and the function \(f_3(x)\) will be decreasing, where the conditions \(0 < cx + r \leq bx + s\) and \(bk \geq ca > 0\) will satisfy \(\forall x \geq 0\) iff \(a, c, r, k, b, s > 0\) also, the condition \(\psi_q(cx + r) > 0\) (or \(\psi_q(bx + s) > 0\)) will be satisfied automatically because if \(f(x) = \Gamma_q(x)\) then \(\psi_q(x) = \alpha(x)\) which is an increasing function.

**Lemma 3.2**

1. For all \(0 \leq x \leq y, c \leq b, r \leq s, ac \leq kb\) and \(a, c, r, k, b, s > 0\), we have

\[
\frac{[\Gamma_q(cy + r)]^a}{[\Gamma_q(by + s)]^k} \leq \frac{[\Gamma_q(cx + r)]^a}{[\Gamma_q(bx + s)]^k} \leq \frac{[\Gamma_q(r)]^a}{[\Gamma_q(s)]^k}. \tag{20}
\]

2. For all \(0 \leq x \geq y, c \geq b, r \geq s, ac \geq kb\) and \(a, c, r, k, b, s > 0\), we have

\[
\frac{[\Gamma_q(r)]^a}{[\Gamma_q(s)]^k} \leq \frac{[\Gamma_q(cx + r)]^a}{[\Gamma_q(bx + s)]^k} \leq \frac{[\Gamma_q(cy + r)]^a}{[\Gamma_q(by + s)]^k}. \tag{21}
\]

### 3.3 Inequalities involving the Riemann zeta function

The gamma function and the Riemann function are connected with relation [2]

\[
\zeta(x + 1)\Gamma(x + 1) = \int_0^\infty \frac{t^x}{e^t - 1} \, dt, \quad x > 0,
\]

where the Riemann zeta function is defined by

\[
\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad x > 1.
\]

The function \(\zeta(x + 1)\Gamma(x + 1)\) is a positive, differentiable and log-convex function for all \(x > 0\) [8]. Then, we get the following Lemma
Lemma 3.3
1— For all $0 < x \leq y$, $c \leq b$, $r \leq s$, $ac \leq kb$ and $a, c, r, k, b, s > 0$, we have
\[
\frac{[\Gamma(cy + r + 1)\zeta(cy + r + 1)]^a}{[\Gamma(by + s + 1)\zeta(by + s + 1)]^k} \leq \frac{[\Gamma(cx + r + 1)\zeta(cx + r + 1)]^a}{[\Gamma(bx + s + 1)\zeta(bx + s + 1)]^k} \leq \frac{[\Gamma(r + 1)\zeta(r + 1)]^a}{[\Gamma(s + 1)\zeta(s + 1)]^k}
\]
(22)

2— For all $0 < x \leq y$, $c \geq b$, $r \geq s$, $ac \geq kb$ and $a, c, r, k, b, s > 0$, we have
\[
\frac{[\Gamma(r + 1)\zeta(r + 1)]^a}{[\Gamma(s + 1)\zeta(s + 1)]^k} \leq \frac{[\Gamma(cx + r + 1)\zeta(cx + r + 1)]^a}{[\Gamma(bx + s + 1)\zeta(bx + s + 1)]^k} \leq \frac{[\Gamma(cy + r + 1)\zeta(cy + r + 1)]^a}{[\Gamma(by + s + 1)\zeta(by + s + 1)]^k}.
\]
(23)

3.3.1 Some special cases

By using (14), we obtain
\[
\frac{[\Gamma(cx + r + 1)\zeta(cx + r + 1)]^a}{[\Gamma(bx + s + 1)\zeta(bx + s + 1)]^k} \leq \frac{[\Gamma(r + 1)\zeta(r + 1)]^a}{[\Gamma(s + 1)\zeta(s + 1)]^k},
\]
(24)

for all $x > 0$, $0 < c \leq b$, $0 \leq r \leq s$, $0 < ac \leq kb$.

Also, (22) and (24) give us that
\[
\frac{[\Gamma(cy + r + 1)\zeta(cy + r + 1)]^a}{[\Gamma(by + s + 1)\zeta(by + s + 1)]^k} \leq \frac{[\Gamma(r + 1)\zeta(r + 1)]^a}{[\Gamma(s + 1)\zeta(s + 1)]^k},
\]
(25)

for all $0 < x \leq y$, $0 < c \leq b$, $0 \leq r \leq s$, $0 < ac \leq kb$.

If we put $y = c = 1$ and $r = s = 0$ in (25), then we get
\[
\frac{[\pi^2]^a}{[\Gamma(b + 1)]^k} \leq \frac{[\zeta(x + 1)]^a}{[\zeta(bx + 1)]^k}, \quad \forall \ 0 < x \leq 1; \ 1 \leq b; \ 0 < a \leq kb.
\]
(26)

where $\Gamma(2) = 1$ and $\zeta(2) = \frac{\pi^2}{6}$. Now, put $a = k = 1$ in (26), we obtain
\[
\zeta(bx + 1) \leq \frac{\zeta(x + 1)\zeta(b + 1)\Gamma(b + 1)}{\pi^2/6}, \quad \forall \ 0 < x \leq 1; \ 1 \leq b.
\]
(27)

Also, put $b = 1$ in (26), to get
\[
\left[\frac{\pi^2}{6}\right]^{a-k} \leq [\zeta(x + 1)]^{a-k}, \quad \forall \ 0 < x \leq 1; \ 0 < a \leq k,
\]
(28)

Hence
\[
\zeta(x + 1) \leq \frac{\pi^2}{6}, \quad \forall \ 0 < x \leq 1
\]
(29)

with equality if $x = 1$. 

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References


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