A Note on Stability of Integrodifferential Equations

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Abstract: We investigate the exponential stability of a nonlinear integrodifferential equation in the form
\[ \frac{dx(t)}{dt} = \int_{t_0}^{t} K(t,s) f(s,x(s)) ds, \]
\[ x(t_0) = x_0. \]

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1- Introduction

Differential and integral inequalities play a prominent role in the study of existence uniqueness, boundedness stability and other qualitative properties of solutions of differential, integral and integrodifferential equation [see [1-7]]. In [4] Massera established the existence of Liapunov function when a system of ordinary differential equations is uniformly asymptotically stable. Miller [5] has shown the existence of a Liapunov function when the linear system
\[ \frac{dx(t)}{dt} = A(t)x(t) + \int_{t_0}^{t} B(t,s)x(s) ds, \quad x(t_0) = x_0 \]  
(1)
is uniformly asymptotic stable.

In this paper we investigate the existence of a Liapunov function when the nonlinear integrodifferential system
\[ \frac{dx(t)}{dt} = \int_{t_0}^{t} K(t,s)f(s,x(s)) ds, \quad x(t_0) = x_0 \]  
(2)
is generalized exponentially stable. In the above equations \( A(t), B(t,s) \) and \( K(t,s) \) are \( n \times n \) matrices defined for \( t_0 \leq t < \infty \) and \( t_0 \leq s \leq t < \infty, t_0 \geq 0 \).

Let \( J \) denote \([t_0, \infty)\) and \( R_+ \) denote half line \([0, \infty)\).
Definition 1: A function \( g \in [R,R] \) is said to belong to the class \( \mathcal{K} \) if \( g(0) = 0 \), \( g(t) \) is monotonic increasing in \( t \). For a vector \( \mathbf{x} \in R^n \), the norm of \( \mathbf{x} \) is defined by \( \|\mathbf{x}\| = \sum_{i=1}^{n}|x_i| \). For a matrix \( A \), the norm is defined by \( \|A\| = \sum_{i,j=1}^{n}|a_{ij}| \), \( S(r) \) denotes the open sphere \( S(r) = \{\mathbf{x} \in R^n : \|\mathbf{x}\| < r\} \).

Consider the integrodifferential equation
\[
\frac{d}{dt} R(t) = A(t)R(t) + \int_{t_0}^{t} B(t,s)R(s)\,ds , \quad R(t_0) = I
\]
where \( I \) is the identity matrix. The proof of the lemma stated below is similar to the argument given in [1] for the matrix system
\[
\frac{dR}{dt} = A(t)R(t), \quad R(t_0) = I
\]
and so we omit the proof.

Lemma 1: The matrix equation (3) has a unique solution \( R(t) , t \in J \). If the following condition are satisfied
(i) \( A(t) \) is continuous on \( J \).
(ii) \( \sup \left\{ \int_{t_0}^{t} \|B(u,s)\| \,du : 0 \leq s \leq t < \infty \right\} \) is bounded.

Further, \( \mathbf{x}(t) = R(t)\mathbf{x}_0 \) satisfies (1) uniquely.

Definition 2: The trivial solution of (1) is generalized exponentially asymptotically stable (GEAS) if
\[
\|\mathbf{x}(t),t_0,\mathbf{x}_0\| \leq K(t)\|\mathbf{x}_0\|\exp\left[P(t_0) - P(t)\right]
\]
where \( K(t) > 0 \) is continuous for \( t \in J \), \( P \in \mathcal{K} \) and \( P(t) \to \infty \) as \( t \to \infty \). If \( \mathcal{K} (t) \equiv K > 0 , P(t) = at , \) with \( a > 0 \) we have the exponential stability.

We assume in the following, that the solution of (1) exists in \( S(r) \) and there exists a function \( V(t,x) \) satisfying the following

(i) \( V \in C\left[ J \times S_r, R_+\right] \) and
\[
\|V(t,x) - V(t,y)\| \leq K(t)\|x - y\| , \quad t \in J , \ x,y \in S_r .
\]

(ii) \( \|x\| \leq V(t,x) \leq K(t)\|x\| , \quad (t,x) \in J \times S_r .
\]

(iii) \( D^+V(t,x) \leq -P(t)V(t,x) , \quad (t,x) \in J \times S_r .
\]
**Theorem 1:** Assume that the solution \( x \equiv 0 \) of (1) is GEAS. Further let \( P'(t) \) exist and be continuous for \( t, J \). Then there exists a function \( V(t,x) \) satisfying the above the conditions (i) \( \rightarrow \) (iii).

**Proof:** Denote \( x(t,t_0,x_0) \) by \( x \). Define \( V(t,x) \) by the relation

\[
V(t,x) = \sup_{s \geq 0} \|x(t+s,t,x)\| \exp p(t+s) p(t) \tag{4}
\]

Before proceeding further we let \( \epsilon(t) = \exp p(t) \), and \( d(t) = \exp(-p(t)) \).

From (4) it follows that (ii) is satisfied. Let \( x, y \in S' \), then

\[
V(t,x) - V(t,y) = \left[ \sup_{s \geq 0} \|x(t+s,t,x)\| \epsilon(t+s) d(t) - \sup_{s \geq 0} \|x(t+s,t,y)\| \epsilon(t+s) d(t) \right]
\]

\[
\leq \sup_{s \geq 0} \|x(t+s,t,x) - x(t+s,t,y)\| \epsilon(t+s) dt
\]

From Lemma 1, it now follows that

\[
\sup_{s \geq 0} \|x(t+s,x-y)\| \epsilon(t+s) dt \leq K(t) \|x-y\|
\]

which establishes (i). It is now shown that \( V(t,x) \) is continuous in \( (t,x) \). Let \( \delta > 0 \) be a real number. Then

\[
\left| V(t+\delta,x') - V(t,x) \right| \leq \left| V(t+\delta,x') - V(t+\delta,x) \right| + \left| V(t+\delta,x) - V(t+\delta,x(t+\delta,x)) \right|
\]

The first two terms on the right side of the preceding inequality are small if \( \|x-x'\| \) and \( \delta \) are sufficiently small, because \( V(t,x) \) is Lipschitzian in and \( x(t+\delta,t,x) \) is continuous in \( \delta \). Hence, because of uniqueness of solution we have

\[
\sup_{s \geq 0} \|x(t+s,t,x)\| d(t) e(t+s-\delta) - \sup_{s \geq 0} \|x(t+s,t,x)\| e(t+s) d(t)
\]

Let \( a(\delta) = \sup_{s \geq 0} \|x(t+s,t,x)\| e(t+s-\delta) d(t) \). Then \( a(\delta) \rightarrow a(0) \) as \( \delta \rightarrow 0 \) and so we have

\[
\sup_{s \geq 0} \|x(t+s,t,x)\| e(t+s-\delta) d(t)
\]

which implies that \( V(t,x) \) is continuous in \( (t,x) \). The proof of the theorem completed if (iii) is verified. Consider

\[
D^*V(t,x(t)) = \lim_{h \rightarrow 0} \sup_{s \geq 0} \left[ \frac{1}{h} \left[ \sup_{s \geq 0} \|x(t+s,t,x)\| e(t+s) d(t+h) - \sup_{s \geq 0} \|x(t+s,t,x)\| e(t+s) dt \right] \right]
\]

\[
\leq -p'(t)V(t,x)
\]
By setting \( g(t, x(t)) = A(t) x(t) + \int_{t_0}^{t} B(t, s) x(s) \, ds \), we have
\[
V(t + h, x + h g(t, x)) - V(t, x) \leq K(t) \left\| x(t + h, t, x) - x - h g(t, x) \right\|
+ V(t + h, x(t + h, t, x)) - V(t, x)
\]
and now it easily follows that
\[
D'V(t, x) \leq p'(t) V(t, x)
\]
along the solution of (1).

**Remark:** It is assumed that the solution of (2) lies in \( S_r \).

**Theorem 2:** Let \( x(t) = x(t, t_0, x_0) \) and \( y(t) = y(t, t_0, y_0) \) represent two solutions of (2) passing through \( (t_0, x_0) \), and \( (t_0, y_0) \) respectively. Further assume that the conditions:

(a) \( f(t, x) \) satisfies
\[
\left\| f(t, x) - f(t, y) \right\| \leq L(t, r) \left\| x - y \right\|, \quad x, y \in S_r, \quad t \in J,
\]
where \( L(t, r) > 0 \) such that \( \int_{t_0}^{\infty} L(S, r) \, ds = M \).

(b) \( k(t, x) \) satisfies
\[
\sup_{s \geq 0} \left\{ \int_{s}^{t} K(u, s) \, ds : 0 \leq s \leq t < \infty \right\} = N.
\]

Then
\[
\left\| x(t) - y(t) \right\| \leq \left\| x_0 - y_0 \right\| \exp(MN)
\]

**Proof:** The proof is a straight forward application of a well known integral inequality. Consider
\[
x(t) - y(t) = x_0 - y_0 + \int_{t_0}^{t} \int_{s}^{t} K(u, s) \, du \left( f(s, x(s)) - f(s, y(s)) \right) \, ds
\]
(which follows by change of order of integration) and so we get
\[
\left\| x(t) - y(t) \right\| \leq \left\| x_0 - y_0 \right\| \int_{t_0}^{t} NL \left\| x(s) - y(s) \right\| \, ds
\]
By the usual Gronwall integral inequality we thus obtain
\[
\left\| x(t) - y(t) \right\| \leq \left\| x_0 - y_0 \right\| \exp \left( \int_{t_0}^{t} NL(s, r) \, ds \right)
\]
\[
\leq \left\| x_0 - y_0 \right\| \exp(MN).
\]

**Theorem 3:** Assume that the trivial solution of (2) is GEAS and satisfying conditions (a) and (b) in Theorem 2. Then there exists a function \( V(t, x) \) fulfilling the conditions:
A note on stability of integrodifferential equations

(i) \[ V \in C(J \times S_r, R_r), \quad 0 < r_0 < r. \]

\[ \left| V(t, x) - V(t, y) \right| \leq \exp(MN) \sup_{s \geq 0} \left[ \exp(t + s) d(t) \right] \| x - y \|, \]

(ii) \[ \| x \| \leq V(t, x) \leq K(t) \| x \|, \quad (t, x) \in J \times S_r; \]

(iii) \[ D^+ V(t, x) \leq -p(t)V(t, x), \quad (t, x) \in J \times S_r. \]

**Proof:** We define Liapunov function by

\[ V(t, x) = \sup_{s \geq 0} \left\| x(t + s, t, x) \exp \left[ p(t + s) - p(t) \right] \right\| \]

(5)

\[ \text{where } r_0 = r / M \text{ where } M = \sup \{ K(t) : t \in J \}. \]

This establishes (ii). The statements (i), (iii) of Theorem 3 can be verified as done in Theorem 1 and hence the proof is complete.

**References**


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