Abstract

In this article, variational iteration method, a well-known method for solving functional equations, has been employed to solve linear and nonlinear Fredholm integrodifferential-difference equations. Results are compared with the exact solutions, which reveal that variational iteration method is very effective and convenient.

Mathematics Subject Classification: 47G20

Keywords: variational iteration method; Fredholm integrodifferential-difference equation

1 Introduction

In recent years, variational iteration method [1–4], has been favorably applied to various kinds of problems, for example, autonomous ordinary differential equation [5], Nonlinear partial differential equations with variable coefficients [6], Schrödinger–Kdv, generalized Kdv and shallow water equations [7], Burger’s and coupled Burger’s equations [8], Linear Helmholtz partial differential equation [9], and others [10,11].
In this article, we apply the method to solve the following mth-order linear and nonlinear Fredholm integrodifferential-difference equation [12].

\[ \sum_{k=0}^{m} p_k(x)u^{(k)}(x) + \sum_{r=0}^{n} p^*_r(x)u^{(r)}(x - \tau) = f(x) + \int_{a}^{b} k(x, t)G(u(t - \tau))dt \quad \tau \geq 0, \]

Where \( p_k(x), p^*_r(x), k(x, t) \) and \( f(x) \) are functions defined on \( a \leq x \leq b. \)

To illustrate the method, consider the following general functional equation

\[ Lu(t) + N(t) = g(t), \]  

(1)

Where \( L \) is a linear operator, \( N \) is a non-linear operator and \( g(t) \) is a known analytical function. According to the variational iteration method, we can construct the following correction functional

\[ u_{n+1}(t) = u_n(t) + \int_{0}^{\xi} \lambda(\xi) \{ Lu_n(\xi) + N\bar{u}_n(\xi) - g(\xi) \} d\xi, \]  

(2)

Where \( \lambda \) is a general Lagrange multiplier which can be identified optimally via variational theory, \( u_0 \) is an initial approximation with possible unknowns, and \( \bar{u}_n \) is considered as restricted variation, i.e., \( \delta\bar{u}_n = 0 \)

2 Application

In this section, we present examples of linear and nonlinear Fredholm integrodifferential-difference equation and results will be compared with the exact solutions.

Example 1. Let us consider the second-order linear Fredholm integrodifferential-difference equation

\[ u^*(x) + xu'(x) + xu(x) + u'(-1) + u(x - 1) = e^{-x} + e + \int_{-1}^{0} t u(t - 1)dt, \]  

(3)

With the following initial conditions

\[ u(0) = 1, \quad u'(0) = -1. \]

The exact solution is \( u(x) = e^{-x}. \)

In the view of the variational iteration method, we construct a correction functional in the following form

\[ u_{n+1}(x) = u_n(x) + \int_{0}^{\xi} \lambda(\xi) \{ u^*_n(\xi) + \tilde{F}[u_n(\xi)] \} d\xi, \]  

(4)
where

\[ F\left[u(x)\right] = x u'(x) + x u(x) + u'(x - 1) + u(x - 1) - e^{-x} - e - \int_{-1}^{0} u(t - 1) dt. \]

\( \bar{F}\left[u_0(\xi)\right] \) is considered as restricted variation, i.e., \( \delta \bar{F}\left[u_0(\xi)\right] = 0. \)

To find the optimal \( \lambda(s)\), calculation variation with respect to \( u_n\), we have the following stationary conditions

\[
\begin{align*}
\lambda'(\xi) &= 0, \\
1 - \lambda'(\xi)|_{\xi=x} &= 0, \\
\lambda(\xi)|_{\xi=x} &= 0.
\end{align*}
\]  

(5)

The Lagrange multiplier, therefore, can be identified as

\[ \lambda = \xi - x. \]

(6)

Substituting the identified multiplier into Eq(4), we have the following iteration formula

\[ u_{n+1}(x) = u_n(x) + \int_{0}^{\xi} (\xi - x) \left\{ u''_n(\xi) + \bar{F}[u_n(\xi)] \right\} d\xi, \]

(7)

We begin with

\[ u_0(x) = 1 - x. \]

Now suppose that \( u(x) \approx u_3(x)\). Some numerical results of these solutions are presented in Table1.
Example 2. Consider the following third–order linear Fredholm integro-differential-difference equation with boundary conditions

\[ u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 0. \]

With the exact solution \( u(x) = \sin x \).

\[
u'''(x) - xu'(x) + u''(x) - xu(x) = -(x+1)(\sin(x-1) + \cos(x)) - \cos 2 + 1 + \int_{-1}^{1} u(t-1) dt,
\]

We can construct the following correction functional

\[
u_{n+1}(x) = u_n(x) + \int_0^1 \lambda(\xi) \left[ u''(\xi) + F[u_n(\xi)] \right] d\xi,
\]

Where

\[
F[u(\xi)] = -xu'(x) + u''(x) - xu(x) + (x+1)(\sin(x-1) + \cos(x)) + \cos 2 - 1 - \int_{-1}^{1} u(t-1) dt.
\]

This yields the stationary conditions

\[
\lambda'''(\xi) = 0,
1 + \lambda''(\xi) \bigg|_{\xi=x} = 0,
\lambda'(\xi) \bigg|_{\xi=x} = 0,
\lambda(\xi) \bigg|_{\xi=x} = 0.
\]

This in turn gives

\[
\lambda = -\frac{1}{2} (\xi - x)^2.
\]
Substituting this value of the Lagrange multiplier into the functional (9) gives

\[ u_{n+1}(x) = u_n(x) - \frac{1}{2} \int_0^x (\xi - x)^2 \{ u_n^{(\nu)}(\xi) + F[u_n(\xi)] \} d\xi, \quad (12) \]

We select the initial value \( u_0(x) = x \).

Now suppose that \( u(x) \approx u_j(x) \), some numerical results of these solutions are presented in Table 2.

Table 2

<table>
<thead>
<tr>
<th>x</th>
<th>( u_{exact}(x) )</th>
<th>( u_j(x) )</th>
<th>Absolute error</th>
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<td>0</td>
<td>0</td>
</tr>
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</table>

**Example 3.** Let us consider the third-order nonlinear Fredholm integro-differential-difference equation

\[ u''''(x) + \frac{1}{2} u''(x) + xu'(x) + 2u'(x-1) + \frac{1}{2} xu(x) + u(x-1) = e + \int_{-1}^0 t u^2(t-1) dt, \quad (13) \]

With the following initial conditions

\[ u(0) = 1, \quad u'(0) = -\frac{1}{2}, \quad u''(0) = \frac{1}{4}. \]

The exact solution is \( u(x) = e^{-\frac{x^2}{2}} \).

We can construct the following correction functional

\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \{ u_n^{(\nu)}(\xi) + F[u_n(\xi)] \} d\xi, \quad (14) \]

where
\[ F[u(\xi)] = \frac{1}{2} u''(x) + x u'(x) + 2 u'(x-1) + \frac{1}{2} x u(x) + u(x-1) - e - \int_1^0 t u^2(t-1) dt. \]

To solve Eq. (13) by means of He’s VIM; we construct a correction functional (see (12)),

\[ u_{n+1}(x) = u_n(x) - \frac{1}{2} \int_0^x (\xi - x)^2 \left( u_n''(\xi) + F\left[u_n(\xi)\right]\right) d\xi, \]

We begin with

\[ u_0(x) = 1 - \frac{1}{2} x + \frac{1}{8} x^2, \]

Now suppose that \( u(x) \approx u_3(x) \), some numerical results of these solutions are presented in Table 3.

### Table 3

<table>
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<tr>
<th>x</th>
<th>( u_{exact}(x) )</th>
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</tbody>
</table>

### 3 Conclusions

In this paper, He's Variational iteration method has been used for finding the solutions for linear and nonlinear Fredholm integro-differential-difference equations.

In Examples 1, 2 and 3, we derive very good approximations to the solutions, the obtained solution shows that the method is a vary convenient and effective to solve wide classes of problems.

The computations associated with the examples in this paper were performed using Maple 10.
References


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