On $W\beta-I^*P$-Open Sets via Pre Local Functions

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Abstract

The aim of this paper is to introduce the new concepts namely, $W\beta-I^*P$-open set, $\beta-I^*P$-open set, $\beta-I^*$-open set, weakly-$\beta-I^*P$-open set, $\beta$-IP-open set and $\beta$-IP$^*$-open set. Some properties and several characterizations are investigated. The relationships between these classes of sets are also obtained.

Mathematics Subject Classification: 54C08, 54A20

Keywords: $W\beta-I^*P$-open set, $\beta-I^*P$-open set, $\beta-I^*$-open set, weakly-$\beta-I^*P$-open set, $\beta$-IP-open set, $\beta$-IP$^*$-open set

1. Introduction and Preliminaries

In 1983, M.E.Abd El Monsef, S.N.EL-Deeb and R.A.Mahmoud[1], introduced $\beta$-open sets and $\beta$-continuous mapping in general topology. In 1992, Jankovic and Hamlett [6] introduced the notion of I - open sets in topological spaces. El - Monsef [2], investigated I -open sets and I - continuous functions. In 1996, Dontchev [4] introduced the notion of pre - I - open sets and obtained a decomposition of I -continuity. In 2002 E.Hatir and T.Noiri[7] introduced the concepts of $\beta$-I-open and $\beta$-I- continuous function. In this paper, the notions of pre local functions,$W\beta-I^*P$-open set, $\beta-I^*P$-open set, $\beta$-IP-open set, $\beta$-IP$^*$-open set, weakly-$\beta$-I$^*P$-open set, $\beta$-IP-open set and $\beta$-IP$^*$-open sets are introduced. The fundamental properties of such functions are studied. Throughout this paper, $\text{cl}(A)$ and $\text{int}(A)$ denote the closure of $A$ and the interior of $A$ respectively. An ideal topological space denoted by $(X, \tau, I)$ is a topological space $(X, \tau)$ with an ideal $I$ on $X$. For a subset $A$ of $X$, $A^*(I) = \{x \in X : U \cap A \notin I$ for each neighbourhood $U$ of $x\}$ is called the local function of $A$ with respect to $I$ and $\tau$. For every ideal topological space $(X, \tau, I)$, there exists a topology $\tau^*(I)$, finer...
than $\tau$, generated by the base $B(\tau, I) = \{U \setminus G : U \in \tau \text{ and } G \in I\}$. Additionally, $\text{cl}^* (A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$. If $(X, \tau, I)$ is a space, we denote by $\tau^* P(I)$ the topology on $X$ generated by the subbasis $\{U \setminus E : U \in PO(X) \text{ and } E \in I\}$. The closure operator in $\tau^* P(I)$ denoted by $\text{cl}^* P$, can be described as follows; for $A \subset X$, $\text{cl}^* P(A) = A \cup A^* P(I)$.

**Definition 1.1.** A subset $S$ of a space $(X, \tau)$ is said to be pre-open [5] (semi-open [7], $\alpha$-open [9], $b$-open [10], $\beta$-open [1]) if $S \subset \text{int(} \text{cl}(S)\text{)}(S \subset \text{cl(} \text{int}(S)\text{)}), \text{S} \subset \text{int(} \text{cl(} \text{int}(S)\text{)}\text{)} \cup \text{int(} \text{cl}(S)\text{)), } S \subset \text{cl(} \text{int}(S)\text{)}\text{)}$).

**Definition 1.2.** A subset $S$ of an ideal topological space $(X, \tau, I)$ is said to be pre-I-open [4](semi-I-open [7], $\alpha$-I-open [7], $\beta$-I-open [7], $b$-I-open [10], weakly-semi-I-open [8]) if $S \subset \text{int(} \text{cl}(S)\text{)}(S \subset \text{cl(} \text{int}(S)\text{)}), S \subset \text{int(} \text{cl}(S)\text{)}), S \subset \text{cl(} \text{int}(S)\text{)} \cup \text{int(} \text{cl}(S)\text{)), S \subset \text{cl}(S)\text{)}\text{)}$).

2.$W$-$\beta$-$I^P$-open set

**Definition 2.1.** Given a space $(X, \tau, I)$, a set operator $(\ast)^P : P(X) \to P(X)$ called the pre local function of $I$ with respect to $\tau$ is defined as follows; for $A \subset X$, $A^*(\tau, I) = \{x \in X : U_x \cap A \notin I\}$, for every $U_x \in PN(x)$, where $PN(x) = \{U \in PO(X) : x \in U\}$. When there is no ambiguity, we will simply write $A^P(\tau, I)$ or $A^*(\tau, I)$.

**Definition 2.2.** A subset $S$ of a space $(X, \tau, I)$ is said to be $W$-$\beta$-$I^P$-open ($\beta$-$I^*\text{-open}, \beta$-$I^P\text{-open}, \text{weakly } \beta$-$I^*\text{-open}, \beta$-$IP^*\text{-open}, \beta$-$IP\text{-open}, \text{pre-}I^P\text{-open}, \text{semi-}I^P\text{-open, } \alpha$-$I^P\text{-open} [3]$) if $S \subset \text{cl}(S)(S \subset \text{cl}(S)), S \subset \text{cl}(S)(S \subset \text{cl}(S)), S \subset \text{cl}(S)(S \subset \text{cl}(S)), S \subset \text{cl}(S)(S \subset \text{cl}(S))\}$.

**Theorem 2.3.** Let $(X, \tau, I)$ be a space and $A \subset X$, then the following hold: 1) Every semi-$I^*\text{-open set is } W\beta$-$I^P\text{-open set.}$. 2) Every pre-$I^P\text{-open set is } W\beta$-$I^P\text{-open set.}$. 3) Every $W\beta$-$I^P\text{-open set is } \beta$-$I\text{-open set.}$. 4) Every $\alpha$-$I^P\text{-open set is } W\beta$-$I^P\text{-open set.}$

Proof. It is obvious.

**Remark 2.4.** The converses are not true in general is shown by the following example.

**Example 2.5.** Let $X = \{a, b, c, d\}, \tau = \{X, \phi\}, \{c, d\}, \{b, c, d\}, I = \{\phi\}, \{d\}$ and $A = \{a, c, d\}$. Then $A$ is $W\beta$-$I^P\text{-open but not } \alpha$-$I^P\text{-open.}$
Proposition 2.6. For a subset of an ideal topological space, the following conditions hold. 1) Every $W\beta \Gamma^P$-open set is $\beta \Gamma^P$-open set. 2) Every $W\beta$ $I^P$-open set is weakly-$\beta I^*P$-open set. 3) Every $\beta I^P$-open set is $\beta$-$I^*$-open set. 4) Every $\beta$-$I^*P$-open set is $\beta I^P$-open set. 5) Every $\beta - I^*$-open set is weakly-$\beta -I^*P$-open set. 6) Every weakly-$\beta I^*P$-open set is weakly-semi-$I$-open set. 7) Every $\beta I^*$-open set is $\beta -I$-open set. 8) Every $\beta I^*$-open set is $\beta \Gamma^P$-open set. 9) Every $\beta I^*$-open set is $\beta I^*$-open set. 10) Every weakly semi-$I$-open set is $\beta$-$I$-open set. 11) Every $\beta I^*$-open set is $\beta$-open set.

Proof. It is obvious.

Remark 2.7. The results in Proposition 2.16. is shown in the following figure.

Proposition 2.8. Let $S$ be a b-$I$-open set such that int $S = \phi$. Then $S$ is $\beta$-$I$-open set.

Proof. Since $S \subset cl^*(int(S)) \cup int(cl^*(S)) = cl^*(\phi) \cup int(cl^*(S)) = int(cl^*(S)) \subset cl(int(cl^*(S)).$

By $W\beta I^P O(X, \tau)$ we denote the family of all $W\beta I^P$-open sets of space $(X, \tau, I)$.

Lemma[3] 2.9. Let $A$ and $B$ be subsets of space $(X, \tau, I)$ then, 1) If $A \subset B$, then $A^*P \subset B^*P$. 2) If $U \in PO(X)$, then $U \cap A^*P \subset (U \cap A)^*P$. 3) $A^*P$ is pre-closed in $(X, \tau)$.

Proof. Obvious.

Theorem 2.10. Let $(X, \tau, I)$ be an ideal topological space and $A, B$ are subsets of $X$. 1) If $U_a \in W\beta I^P O(X, \tau)$ for each $a \in \Delta$ then $\cup \{U_a : a \in \Delta\} \in W\beta I^P O(X, \tau)$. 2) If $A \in W\beta I^P O(X, \tau) and B \in PO(X)$ then $A \cap B \notin W\beta I^P O(X, \tau)$.

Proof. 1) Let $U_a \in W\beta I^P O(X, \tau)$, we have $U_a \subset cl^P(int(cl^P(U_a)))$ for every $a \in \Delta$. $\cup_{a \in \Delta} U_a \subset cl^P(int(cl^P(U_a)))$

\[ \substack{\subset \cup_{a \in \Delta}(int(cl^P(U_a))) \cup (int(cl^P(U_a)))^*P \\
\subset \{\cup_{a \in \Delta}(int(cl^P(U_a))) \cup (\cup_{a \in \Delta}(int(cl^P(U_a))))^*P \}
\subset \{(\cup_{a \in \Delta} int(U_a \cup (U_a)^*P)) \cup (\cup_{a \in \Delta} int(U_a \cup (U_a)^*P))\}
\subset \{(\cup_{a \in \Delta} int(U_a) \cup (\cup_{a \in \Delta} (U_a)^*P))\}
\subset \{(\cup_{a \in \Delta} int(U_a)) \cup (U_a)^*P\) \cup (int((\cup_{a \in \Delta} U_a) \cup (\cup_{a \in \Delta} (U_a)^*P))\}
\subset \{(\cup_{a \in \Delta} int(U_a)) \cup (U_a)^*P\) \cup (int((\cup_{a \in \Delta} U_a) \cup (\cup_{a \in \Delta} (U_a)^*P))\)
\]
\[
\{ \text{cl}^P(\text{int}(\bigcup_{a \in \Delta} U_a) \cup (\bigcup_{a \in \Delta} U_a)^*P) \}
\]

\[
= \text{cl}^P(\text{int}(\text{cl}^P(\bigcup_{a \in \Delta} U_a)))
\]

which implies, \( \bigcup_{a \in \Delta} U_a \in W\beta P\text{O}(X, \tau) \).

2) Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \phi, \{b\}, \{c, d\}, \{b, c, d\}\} \), \( I = \{\phi, \{d\}\} \) then \( A = \{a, c, d\} \) is \( W\beta P\text{I}^P \)-open set and \( \{d\} \) is a preopen set but \( A \cap \text{PO}(X) = \{d\} \) is not a \( W\beta P\text{I}^P \)-open set.

**Definition 2.11.** A subset \( S \) is said to be \( W\beta P\text{I}^P \)-closed if its complement is \( W\beta P\text{I}^P \) open.

**Theorem 2.12.** A subset \( A \) of a space \((X, \tau, I)\) is \( W\beta P\text{I}^P \)-closed if and only if \( \text{int}^P(\text{cl}(\text{int}^P(A))) \subseteq A \).

Proof. Let \( A \) be \( W\beta P\text{I}^P \)-closed set of \((X, \tau, I)\). Then \( X - A \) is \( W\beta P\text{I}^P \) open and hence \( X - A \subseteq \text{cl}^P(\text{int}(\text{cl}^P(X - A))) = A \supseteq \text{int}^P(\text{cl}(\text{int}^P(A))). \)

There- \( \text{int}^P(\text{cl}(\text{int}^P(A))) \subseteq A. \) Conversely, let \( \text{int}^P(\text{cl}(\text{int}^P(A))) \subseteq A \) then \( X - A \subseteq X - \text{int}^P(\text{cl}(\text{int}^P(A))) = \text{cl}^P(\text{int}(\text{cl}^P(X - A))). \) (i.e) \( X - A \) is \( W\beta P\text{I}^P \)-open. Therefore, \( A \) is \( W\beta P\text{I}^P \)-closed.

**Remark 2.13.** For a subset \( A \) of a space \((X, \tau, I)\) we have \( X - \text{int}(\text{cl}^P(\text{int}(A))) \neq \text{cl}^P(\text{int}(\text{cl}^P(X - A))) \) is shown by the following.

**Example 2.14.** Let \( X = \{a, b, c\}, \tau = \{\phi, X, \{a\}\} \), \( I = \{\phi, \{a\}\} \) and \( A = \{b, c\}. \)

Then, \( X - \text{int}(\text{cl}^P(\text{int}(A))) = X, \) but \( \text{cl}^P(\text{int}(\text{cl}^P(X - A))) = \{a\}. \) Hence \( X - \text{int}(\text{cl}^P(\text{int}(A))) \neq \text{cl}^P(\text{int}(\text{cl}^P(X - A))). \)

**Theorem 2.15.** If \( A \subseteq (X, \tau, I) \) is \( W\beta P\text{I}^P \)-closed then \( \text{int}(\text{cl}^P(\text{int}(A))) \subseteq A. \)

Proof. Obvious.

**Corollary 2.16.** Let \( A \) be a subset of a space \((X, \tau, I)\) such that \( X - \text{int}(\text{cl}^P(\text{int}(A))) = \text{cl}^P(\text{int}(\text{cl}^P(X - A))). \) Then \( A \) is \( W\beta P\text{I}^P \)-closed if and only if \( \text{int}(\text{cl}^P(\text{int}(A))) \subseteq A. \)

Proof. This is an immediate consequence of Theorem 2.15.

**Definition 2.17.** A subset \( A \) of a space \((X, \tau, I)\) is called 1) Strong \( W\beta P\text{I}^P \)-set if \( \text{cl}^P(\text{int}(\text{cl}^P(A))) = \text{int}(A). \) 2) \( W\beta P\text{I}^P \)-set if \( \text{cl}^P(\text{int}(A)) = \text{int}(A). \)

**Definition 2.18.** A subset of an ideal topological space \((X, \tau, I)\) is called 1) Strong \( W\beta P\text{I}_I^P \)-set if \( A = U \cap V, \) where \( U \in \tau \) and \( V \) is Strong \( W\beta P\text{I}^P \)-set. 2) \( W\beta P\text{I}_I^P \)-set if \( A = U \cap V, \) where \( U \in \tau \) and \( V \) is \( W\beta P\text{I}^P \)-set.

**Remark 2.19.** For a subset \( A \) of a space \((X, \tau, I)\) we have a) Every Strong \( W\beta P\text{I}^P \)-set is \( W\beta P\text{I}^P \)-set. b) Every Strong \( W\beta P\text{I}_I^P \)-set is \( W\beta P\text{I}_I^P \) - set.
c) Every open set is Strong $W\beta P_{I,*P}$ set.

Proof. a) Let $A$ be a Strong $W\beta P_{I,*P}$-set then $\text{cl}^{*P}(\text{int}(\text{cl}^{*P}(A))) = \text{int}(A)$. $\text{cl}^{*P}(\text{int}(A)) \subseteq \text{cl}^{*P}(\text{int}(\text{cl}^{*P}(A))) \subset \text{int}(A)$, (i.e) (by using definition), But $\text{cl}^{*P}(\text{int}(A)) = \text{int}(A) \cup (\text{int}(A))^{*P}$ and so $\text{cl}^{*P}(\text{int}(A)) \cup (\text{int}(A))^{*P}$ is Strong $W\beta P_{I,*P}$-open. Then by Remark 2.19 c), every open set is Strong $W\beta P_{I,*P}$-set. Therefore, $\text{cl}^{*P}(\text{int}(A)) = \text{int}(A)$. (i.e) $A$ is a Strong $W\beta P_{I,*P}$-set. b) Every Strong $W\beta P_{I,*P}$-set is $W\beta P_{I,*P}$-set. Let $A$ be a Strong $W\beta P_{I,*P}$-set, then $A = U \cap V$, where $U \in \tau$ and $V$ is a Strong $W\beta P_{I,*P}$-set (by using (a) above) $V$ is $W\beta P_{I,*P}$-set. Thus $A$ is a $W\beta P_{I,*P}$ set. c) Every open set is Strong $W\beta P_{I,*P}$-set. Let $A$ be an open set Then $A = A \cap X$, where $A \in \tau$ and $X$ is a Strong $W\beta P_{I,*P}$-set Therefore, $A$ is a Strong $W\beta P_{I,*P}$-set.

Proposition 2.20. For a subset $(X, \tau, I)$ the following conditions are equivalent a) $A$ is open. b) $A$ is $W\beta I_{*P}$-open and Strong $W\beta P_{I,*P}$-open. c) $A$ is semi-$I_{*P}$-open and $W\beta P_{I,*P}$-open.

Proof. a) $\rightarrow$ b) Let $A$ be open set, then by Remark 2.19(c), $A$ is Strong $W\beta P_{I,*P}$-set Given $A = \text{int} A$ which implies, $A = \text{int} \subset \text{int}(\text{cl}^{*P}(A)) \subset \text{cl}^{*P}(\text{int}(\text{cl}^{*P}(A)))$ Therefore, $A$ is $W\beta I_{*P}$-open. b) $\rightarrow$ a) Let $A$ be $W\beta I_{*P}$-open and Strong $W\beta P_{I,*P}$-set Then $A = U \cap V$, where $U \in \tau$, $V$ is Strong $W\beta P_{I,*P}$-set, and $A \subset \text{cl}^{*P}(\text{int}(\text{cl}^{*P}(A)))$ Therefore, $A \subset \text{cl}^{*P}(\text{int}(\text{cl}^{*P}(U \cap V)))$.

Since $A \subset U \cap A$, we get $A \subset U \cap [\text{cl}^{*P}(\text{int}(\text{cl}^{*P}(U))) \cap \text{cl}^{*P}(\text{int}(\text{cl}^{*P}(V)))]$ But $U = \text{int} U \subset \text{int}(\text{cl}^{*P}(U)) \subset \text{cl}^{*P}(\text{int}(\text{cl}^{*P}(U)))$ and $\text{cl}^{*P}(\text{int}(\text{cl}^{*P}(V))) \subset \text{int}(\text{cl}^{*P}(V)) = \text{int} V$ we get $A \subset U \cap \text{int} V = \text{int} U \cap \text{int} V = \text{int}(U \cap V) = \text{int} A$. Thus $A$ is open. a) $\rightarrow$ c) Let $A$ be open, then $A = \text{int}(A) \subset \text{cl}^{*P}(\text{int}(A))$. Therefore $A$ is semi-$I_{*P}$-open. Then by Remark 2.19 c), every open set is $W\beta P_{I,*P}$-set. Therefore, $A$ is both semi-$I_{*P}$-open and $W\beta P_{I,*P}$-set. c) $\rightarrow$ a) Let $A$ be semi-$I_{*P}$-open and $W\beta P_{I,*P}$-set Then $A \subset \text{cl}^{*P}(\text{int}(A))$ and $A = U \cap V$, where $U \in \tau$ and $V$ is $W\beta P_{I,*P}$-set. $A \subset U \cap [\text{cl}^{*P}(\text{int} U) \cap \text{cl}^{*P}(\text{int} V)] \subset U \cap \text{cl}^{*P}(\text{int}(\text{cl}^{*P}(V))) = U \cap \text{int} V = \text{int} A$. Therefore, $A$ is open.

References


[3] I.Arockia Rani and A.A.Nithya, on $\alpha$-$I_{*P}$-open sets via pre local functions, communicated.


Received: June, 2010