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On Symmetric Block Design with Parameters (189,48,12)

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Abstract

In this paper is proved that:
A). Up to isomorphism and duality there are exactly 27 orbital structures for a putative symmetric block design $\mathcal{D}$ with parameters (189,48,12), constructed by Elementary Abelian Group G_{27}, using the collineation $\mu = (1)(2,3,4)(5,6,7)$ of order three, that commutes with collineation $|\rho| = 27$ of group G_{27}.
B). Up to isomorphism and duality there are exactly two orbital structures for a putative symmetric block design $\mathcal{D}$ with parameters (189,48,12), constructed using the collineation ρ of order 47, which operates on $\mathcal{D}$ with one fix point (block), whereas on the other points (blocks) operates transitively.

Mathematics Subject Classification: 05B05.

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1 Introduction

We assume that the reader is familiar with the basic facts of design theory. For introductory material see for instance [1], [2] and [5].
Among symmetric block designs of square order, a study of symmetric block designs of order 36 is of a particular interest. There are 18 possible parameters (v, k, λ) for symmetric designs of order 36, but until now only a few results are known. Among these are the parameters of the projective plane of order 36, which is the first projective plane of square order, the existence of which is still undecided.

Due to the fact that symmetric designs of order 36 have a big number of points (blocks), the study of sporadic cases is very difficult, except, possibly, when the existence of a collineation group is assumed.
A few methods for the construction of symmetric block designs are known and all of them have shown to be effective in certain situations. Here, we shall use the method of tactical decompositions, assuming that a certain automorphism group acts on the design we want to construct, suggested and used by Zvonimir Janko. [7] (see also [6]).

2 Main Results:

A). Let G_{27} be an Elementary Abelian Group and ρ its element of order 27. It is clear that the collineation ρ operates in the symmetric block design $\mathcal{D}$ as

$$\rho = (1_0, 1_1, 1_2, \ldots, 1_{26})(2_0, 2_1, 2_2, \ldots, 2_{26}) \ldots (7_0, 7_1, 7_2, \ldots, 7_{26}).$$

The collineation $\mu = (1)(2,3,4)(5,6,7)$ of order three commutes with collineation ρ. Now we are going to construct the orbital structures of $\mathcal{D}$ using Elementary Abelian Group G_{27} and collineation $\mu = (1)(2,3,4)(5,6,7)$.
Let $l_1 = 1_{a_1} 2_{a_2} 3_{a_2} 4_{a_2} 5_{a_3} 6_{a_3} 7_{a_3}$ be the first μ-invariant block, where the numbers a_1, a_2, a_3 denote the multiplicity of appearance of orbital numbers in the orbital block l_1. Since $k = 12$, we have $a_1 + 3a_2 + 3a_3 = 12$.
The block l_1 must satisfy condition of Hamming length, namely

$$H(l_1) = (|\rho| - 1) \cdot \lambda = 312, \text{i.e.}$$
$$a_1(a_1 - 1) + 3a_2(a_2 - 1) + 3a_3(a_3 - 1) = 312.$$

From the last relation, for the multiplicity of appearance in the block l_1, we have these reductions $0 \leq a_1, a_2, a_3 \leq 18$. In order to reduce isomorphic cases that may appear in the orbital structures at the last stage, without loss of generality, for the first μ-invariant block we may assume that the inequality $a_2 \leq a_3$ hold for the block l_1.
There are exactly two orbital types for the block l_1, that satisfy above conditions:
1) $l_1 = 1_3\ 2_6\ 3_6\ 4_6\ 5_9\ 6_9\ 7_9$
2) $l_1 = 1_{12}\ 2_6\ 3_6\ 4_6\ 5_6\ 6_6\ 7_6$

If we denote $l_2 = 1_{b_1}\ 2_{b_2}\ 3_{b_3}\ 4_{b_4}\ 5_{b_5}\ 6_{b_6}\ 7_{b_7}$ the second ρ–orbital block then the second μ–orbital blocks of design $\mathcal{D}$ are:

$$\begin{aligned} l_2 = &\ 1_{b_1}\ 2_{b_2}\ 3_{b_3}\ 4_{b_4}\ 5_{b_5}\ 6_{b_6}\ 7_{b_7} \\ l_2^{\mu} = l_3 = &\ 1_{b_1}\ 2_{b_4}\ 3_{b_2}\ 4_{b_3}\ 5_{b_7}\ 6_{b_5}\ 7_{b_6} \\ l_2^{\mu^2} = l_4 = &\ 1_{b_1}\ 2_{b_3}\ 3_{b_4}\ 4_{b_2}\ 5_{b_6}\ 6_{b_7}\ 7_{b_5} \end{aligned}$$

Since $k = 12$, $H(l_2) = (|\rho| - 1) \cdot \lambda = 312$ and $|l_i \cap l_j| = 21$, where $i, j = 1, 2, 3$ nd $i \neq j$, we have

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 = 48 \tag{1}$$

and

$$b_1(b_1-1)+b_2(b_2-1)+b_3(b_3-1)+b_4(b_4-1)+b_5(b_5-1)+b_6(b_6-1)+b_7(b_7-1) = 312.$$

From the last relation, for the multiplicity of appearance in the block l_2, we obtain the reductions $0 \leq b_i \leq 18, i = 1, 2, \ldots, 7$.
The game product $S_p(l_1, l_2) = |\rho| \cdot \lambda = 324$ implies relation:

$$a_1b_1 + a_2b_2 + a_2b_3 + a_2b_4 + a_3b_5 + a_3b_6 + a_3b_7 = 324 \tag{2}$$

From (1) and (2) we find $b_1 = 6$. But, due to fact

$$S_p(l_2, l_2^{\mu}) = S_p(l_2, l_2^{\mu})^{\mu^2} = S_p(l_2^{\mu^2}, l_2)^{\mu^2} = S_p(l_2^{\mu}, l_2^{\mu^2}),$$

the multiplicity of appearance must satisfy only $S_p(l_2, l_2^{\mu}) = 324$, i.e.

$$b_1^2 + b_2b_4 + b_3b_2 + b_4b_3 + b_5b_7 + b_6b_5 + b_7b_6 = 324.$$

In order to reduce isomorphic cases that may appear in the orbital structures at the last stage we can use collineations $\eta = (2, 3, 4)$ and $\xi = (5, 6, 7)$ which commute with collineation $\mu = (1)(2, 3, 4)(5, 6, 7)$ and collineation $\zeta = (1)(2)(3, 4)(5)(6, 7)$ which inverts (normalizes) collineation μ.
Collineations η, ξ and ζ allow reductions $b_2 \leq b_3 \leq b_4, b_5 \leq b_6$ and $b_5 \leq b_7$.
Based on above, we have proved that there are exactly 24 orbital structures, of design $\mathcal{D}$, up to second μ–orbit block.
Now we construct the third μ–orbit of blocks of design $\mathcal{D}$ in connection with collineation ρ.
Denoting $l_5 = l_{c_1}2_{c_2}3_{c_3}4_{c_4}5_{c_5}6_{c_6}7_{c_7}$ and considering the action of the collinaetion μ, the third μ–orbit block of design $\mathcal{D}$ is:

$$\begin{array}{rl} l_5 = & l_{c_1}\ 2_{c_2}\ 3_{c_3}\ 4_{c_4}\ 5_{c_5}\ 6_{c_6}\ 7_{c_7} \\ l_5^{\mu} = l_6 = & l_{c_1}\ 2_{c_4}\ 3_{c_2}\ 4_{c_3}\ 5_{c_7}\ 6_{c_5}\ 7_{c_6} \\ l_5^{\mu^2} = l_7 = & l_{c_1}\ 2_{c_3}\ 3_{c_4}\ 4_{c_2}\ 5_{c_6}\ 6_{c_4}\ 7_{c_5} \end{array} \tag{$*$}$$

Multiplicity of appearances $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ of third μ–orbit (*) must satisfy the following

$$c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 = 48 \tag{3}$$

and

$$H(l_5) = (|\rho| - 1) \cdot \lambda = 312$$

i.e.

$$c_1(c_1-1)+c_2(c_2-1)+c_3(c_3-1)+c_4(c_4-1)+c_5(c_5-1)+c_6(c_6-1)+c_7(c_7-1)=312.$$

From the last relation we obtain the reductions $0 \leq c_i \leq 18,\ i=1,2,3,4,5,6,7$. Since $S_p(l_5,l_1)=324$ and $k=48$ we have $c_1=6$.
Blocks of third μ–orbit must satisfy Game Products with each block of second μ–orbit and with μ–fix block and, finally, each two blocks of third μ–orbit also must satisfy Game Product among them.
But, since

$$S_p(l_2,l_5)=S_p(l_2^{\mu},l_5^{\mu})=S_p(l_2^{\mu^2},l_5^{\mu^2}),$$

$$S_p(l_2,l_5^{\mu})=S_p(l_2^{\mu},l_5^{\mu^2})=S_p(l_2^{\mu^2},l_5),$$

$$S_p(l_2,l_5^{\mu^2})=S_p(l_2^{\mu^2},l_5^{\mu})=S_p(l_2^{\mu},l_5),$$

$$S_p(l_5,l_5^{\mu})=S_p(l_5^{\mu},l_5^{\mu^2})=S_p(l_5^{\mu^2},l_5),$$

only the following conditions have to be satisfied:

$$S_p(l_2,l_5)=324, S_p(l_2,l_5^{\mu})=324, S_p(l_2,l_5^{\mu^2})=324 \text{ and } S_p(l_5,l_5^{\mu})=324.$$

Conditions of orbital numbers in the block l_5 and its μ–images yield:

$$b_1c_1+b_2c_2+b_3c_3+b_4c_4+b_5c_5+b_6c_6+b_7c_7=324,$$

$$b_1c_1+b_2c_4+b_3c_2+b_4c_3+b_5c_7+b_6c_5+b_7c_6=324,$$

$$b_1c_1+b_2c_3+b_3c_4+b_4c_2+b_5c_6+b_6c_7+b_7c_5=324,$$

$$c_1^2+c_2c_4+c_3c_2+c_4c_3+c_5c_7+c_6c_5+c_7c_6=324,$$

where $b_1,b_2,b_3,b_4,b_5,b_6,b_7$ take values from 24 solutions of the orbital block l_2.
Based on above conditions the multiplicity of appearance $c_1,c_2,c_3,c_4,c_5,c_6,c_7$ of block l_5 and their μ–figures must satisfy, we have proved that there are exactly 15 orbital structures of symmetric block design (189,48,12) for the first orbital type and 12 orbital structures for the second orbital type.
For the first orbital type, 15 orbital structures are:

$$(1)\ \begin{matrix} 3&6&6&6&9&9&9\\ 6&4&10&10&6&6&6\\ 6&10&4&10&6&6&6\\ 6&10&10&4&6&6&6\\ 9&6&6&6&9&9&3\\ 9&6&6&6&3&9&9\\ 9&6&6&6&9&3&9 \end{matrix} \qquad (2)\ \begin{matrix} 3&6&6&6&9&9&9\\ 6&4&10&10&6&6&6\\ 6&10&4&10&6&6&6\\ 6&10&10&4&6&6&6\\ 9&6&6&6&11&5&5\\ 9&6&6&6&5&11&5\\ 9&6&6&6&5&5&11 \end{matrix} \qquad (3)\ \begin{matrix} 3&6&6&6&9&9&9\\ 6&5&8&11&4&7&7\\ 6&11&5&8&7&4&7\\ 6&8&11&5&7&7&4\\ 9&5&8&5&7&4&10\\ 9&5&5&8&10&7&4\\ 9&8&5&5&4&10&7 \end{matrix}$$

$$(4)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 6 & 6 & 6 & 12 & 6 & 6 & 6 \\ 6 & 12 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 12 & 6 & 6 & 6 & 6 \\ 9 & 6 & 6 & 6 & 11 & 5 & 5 \\ 9 & 6 & 6 & 6 & 5 & 11 & 5 \\ 9 & 6 & 6 & 6 & 5 & 5 & 11 \end{array}$$

$$(5)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 6 & 8 & 8 & 8 & 2 & 8 & 8 \\ 6 & 8 & 8 & 8 & 8 & 2 & 8 \\ 6 & 8 & 8 & 8 & 8 & 8 & 2 \\ 9 & 10 & 4 & 4 & 7 & 7 & 7 \\ 9 & 4 & 10 & 4 & 7 & 7 & 7 \\ 9 & 4 & 4 & 10 & 7 & 7 & 7 \end{array}$$

$$(6)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 9 & 2 & 8 & 8 & 7 & 7 & 7 \\ 9 & 8 & 2 & 8 & 7 & 7 & 7 \\ 9 & 8 & 8 & 2 & 7 & 7 & 7 \\ 6 & 8 & 8 & 8 & 2 & 8 & 8 \\ 6 & 8 & 8 & 8 & 8 & 2 & 8 \\ 6 & 8 & 8 & 8 & 8 & 8 & 2 \end{array}$$

$$(7)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 9 & 3 & 6 & 9 & 5 & 8 & 8 \\ 9 & 9 & 3 & 6 & 8 & 5 & 8 \\ 9 & 6 & 9 & 3 & 8 & 8 & 5 \\ 6 & 10 & 7 & 7 & 3 & 9 & 6 \\ 6 & 7 & 10 & 7 & 6 & 3 & 9 \\ 6 & 7 & 7 & 10 & 9 & 6 & 3 \end{array}$$

$$(8)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 9 & 3 & 6 & 9 & 5 & 8 & 8 \\ 9 & 9 & 3 & 6 & 8 & 5 & 8 \\ 9 & 6 & 9 & 3 & 8 & 8 & 5 \\ 6 & 9 & 9 & 6 & 3 & 6 & 9 \\ 6 & 6 & 9 & 9 & 9 & 3 & 6 \\ 6 & 9 & 6 & 9 & 6 & 9 & 3 \end{array}$$

$$(9)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 9 & 3 & 6 & 9 & 6 & 6 & 9 \\ 9 & 9 & 3 & 6 & 9 & 6 & 6 \\ 9 & 6 & 9 & 3 & 6 & 9 & 6 \\ 6 & 9 & 6 & 9 & 3 & 9 & 6 \\ 6 & 9 & 9 & 6 & 6 & 3 & 9 \\ 6 & 6 & 9 & 9 & 9 & 6 & 3 \end{array}$$

$$(10)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 9 & 3 & 6 & 9 & 6 & 6 & 9 \\ 9 & 9 & 3 & 6 & 9 & 6 & 6 \\ 9 & 6 & 9 & 3 & 6 & 9 & 6 \\ 6 & 10 & 7 & 7 & 3 & 6 & 9 \\ 6 & 7 & 10 & 7 & 9 & 3 & 6 \\ 6 & 7 & 7 & 10 & 6 & 9 & 3 \end{array}$$

$$(11)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 9 & 4 & 4 & 10 & 7 & 7 & 7 \\ 9 & 10 & 4 & 4 & 7 & 7 & 7 \\ 9 & 4 & 10 & 4 & 7 & 7 & 7 \\ 6 & 8 & 8 & 8 & 4 & 4 & 10 \\ 6 & 8 & 8 & 8 & 10 & 4 & 4 \\ 6 & 8 & 8 & 8 & 4 & 10 & 4 \end{array}$$

$$(12)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 9 & 4 & 7 & 7 & 4 & 7 & 10 \\ 9 & 7 & 4 & 7 & 10 & 4 & 7 \\ 9 & 7 & 7 & 4 & 7 & 10 & 4 \\ 6 & 11 & 5 & 8 & 4 & 7 & 7 \\ 6 & 8 & 11 & 5 & 7 & 4 & 7 \\ 6 & 5 & 8 & 11 & 7 & 7 & 4 \end{array}$$

$$(13)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 9 & 5 & 5 & 8 & 4 & 7 & 10 \\ 9 & 8 & 5 & 5 & 10 & 4 & 7 \\ 9 & 5 & 8 & 5 & 7 & 10 & 4 \\ 6 & 11 & 5 & 8 & 5 & 8 & 5 \\ 6 & 8 & 11 & 5 & 5 & 5 & 8 \\ 6 & 5 & 8 & 11 & 8 & 5 & 5 \end{array}$$

$$(14)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 9 & 5 & 5 & 8 & 4 & 7 & 10 \\ 9 & 8 & 5 & 5 & 10 & 4 & 7 \\ 9 & 5 & 8 & 5 & 7 & 10 & 4 \\ 6 & 8 & 5 & 11 & 7 & 7 & 4 \\ 6 & 11 & 5 & 8 & 4 & 7 & 7 \\ 6 & 5 & 11 & 8 & 7 & 4 & 7 \end{array}$$

$$(15)\quad \begin{array}{ccccccc} 3 & 6 & 6 & 6 & 9 & 9 & 9 \\ 9 & 6 & 6 & 6 & 3 & 9 & 9 \\ 9 & 6 & 6 & 6 & 9 & 3 & 9 \\ 9 & 6 & 6 & 6 & 9 & 9 & 3 \\ 6 & 12 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 12 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 12 & 6 & 6 & 6 \end{array}$$

For the second orbital type, 12 orbital structures are:

$$(1)\quad \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 2 & 8 & 8 & 8 & 8 & 8 \\ 6 & 8 & 2 & 8 & 8 & 8 & 8 \\ 6 & 8 & 8 & 2 & 8 & 8 & 8 \\ 6 & 8 & 8 & 8 & 2 & 8 & 8 \\ 6 & 8 & 8 & 8 & 8 & 2 & 8 \\ 6 & 8 & 8 & 8 & 8 & 8 & 2 \end{array}$$

$$(2)\quad \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 2 & 8 & 8 & 8 & 8 & 8 \\ 6 & 8 & 2 & 8 & 8 & 8 & 8 \\ 6 & 8 & 8 & 2 & 8 & 8 & 8 \\ 6 & 8 & 8 & 8 & 10 & 4 & 4 \\ 6 & 8 & 8 & 8 & 4 & 10 & 4 \\ 6 & 8 & 8 & 8 & 4 & 4 & 10 \end{array}$$

$$(3)\quad \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 3 & 6 & 9 & 6 & 9 & 9 \\ 6 & 9 & 3 & 6 & 9 & 6 & 9 \\ 6 & 6 & 9 & 3 & 9 & 9 & 6 \\ 6 & 9 & 6 & 6 & 3 & 6 & 9 \\ 6 & 6 & 9 & 9 & 9 & 3 & 6 \\ 6 & 9 & 6 & 9 & 6 & 9 & 3 \end{array}$$

$$(4)\ \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 3 & 6 & 9 & 6 & 9 & 9 \\ 6 & 9 & 3 & 6 & 9 & 6 & 9 \\ 6 & 6 & 9 & 3 & 9 & 9 & 6 \\ 6 & 10 & 7 & 7 & 3 & 9 & 6 \\ 6 & 7 & 10 & 7 & 6 & 3 & 9 \\ 6 & 7 & 7 & 10 & 9 & 6 & 3 \end{array} \quad (5)\ \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 3 & 6 & 9 & 7 & 10 & 7 \\ 6 & 9 & 3 & 6 & 7 & 7 & 10 \\ 6 & 6 & 9 & 3 & 10 & 7 & 7 \\ 6 & 10 & 7 & 7 & 6 & 9 & 3 \\ 6 & 7 & 10 & 7 & 3 & 6 & 9 \\ 6 & 7 & 7 & 10 & 9 & 3 & 6 \end{array} \quad (6)\ \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 4 & 4 & 10 & 8 & 8 & 8 \\ 6 & 10 & 4 & 4 & 8 & 8 & 8 \\ 6 & 4 & 10 & 4 & 8 & 8 & 8 \\ 6 & 8 & 8 & 8 & 10 & 4 & 4 \\ 6 & 8 & 8 & 8 & 4 & 10 & 4 \\ 6 & 8 & 8 & 8 & 4 & 4 & 10 \end{array}$$

$$(7)\ \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 4 & 7 & 7 & 5 & 11 & 8 \\ 6 & 7 & 4 & 7 & 8 & 5 & 11 \\ 6 & 7 & 7 & 4 & 11 & 8 & 5 \\ 6 & 11 & 8 & 5 & 4 & 7 & 7 \\ 6 & 5 & 11 & 8 & 7 & 4 & 7 \\ 6 & 8 & 5 & 11 & 7 & 7 & 4 \end{array} \quad (8)\ \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 4 & 10 & 10 & 6 & 6 & 6 \\ 6 & 10 & 4 & 10 & 6 & 6 & 6 \\ 6 & 10 & 10 & 4 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 4 & 10 & 10 \\ 6 & 6 & 6 & 6 & 10 & 4 & 10 \\ 6 & 6 & 6 & 6 & 10 & 10 & 4 \end{array} \quad (9)\ \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 4 & 10 & 10 & 6 & 6 & 6 \\ 6 & 10 & 4 & 10 & 6 & 6 & 6 \\ 6 & 10 & 10 & 4 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 12 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 12 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 12 \end{array}$$

$$(10)\ \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 5 & 5 & 8 & 5 & 11 & 8 \\ 6 & 8 & 5 & 5 & 8 & 5 & 11 \\ 6 & 5 & 8 & 5 & 11 & 8 & 5 \\ 6 & 11 & 5 & 8 & 7 & 7 & 4 \\ 6 & 8 & 11 & 5 & 4 & 7 & 7 \\ 6 & 5 & 8 & 11 & 7 & 4 & 7 \end{array} \quad (11)\ \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 5 & 8 & 11 & 5 & 8 & 5 \\ 6 & 11 & 5 & 8 & 5 & 5 & 8 \\ 6 & 8 & 11 & 5 & 8 & 5 & 5 \\ 6 & 8 & 5 & 5 & 8 & 11 & 5 \\ 6 & 5 & 8 & 5 & 5 & 8 & 11 \\ 6 & 5 & 5 & 8 & 11 & 5 & 8 \end{array} \quad (12)\ \begin{array}{ccccccc} 12 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 12 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 12 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 12 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 12 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 12 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 12 \end{array}$$

In this way we have proved the following:

Theorem 2.1 *Up to isomorphism and duality there are exactly 27 orbital structures of symmetric block design (189, 48, 12) constructed by Elementary Abelian Group G_{27}, using the collineation $\mu = (1)(2,3,4)(5,6,7)$ of order three that commutes with collineation $|\rho| = 27$ of the group G_{27}.*

Note 2.1 *The actual indexing of these 27 orbital structures in order to produce an example is still an open problem.*

B. Denote $\mathcal{D}$ the symmetric block design with parameters (189,48,12). Since $v = 1 + 4 \cdot 47$ and $k - 1 = 47$, in order to construct the symmetric block design $\mathcal{D}$ we use the collineation ρ of order 47, which fixes one point of $\mathcal{D}$ while in other points (blocks) operates transitively. So, acting of ρ on $\mathcal{D}$ is:

$$\rho = (\infty)(1_0, 1_1, 1_2, \ldots, 1_{46})(2_0, 2_1, 2_2, \ldots, 2_{46})(3_0, 3_1, 3_2, \ldots, 3_{46})(4_0, 4_1, 4_2, \ldots, 4_{46})$$

where ∞ is fixes point, while 1,2,3,4,5,6 are six ρ–orbit points. The ρ–fix block of $\mathcal{D}$ has the form as follow

$$l_1 = \infty\ 1_{47}.$$

The second ρ–orbit block l_2 of design $\mathcal{D}$, constructed by collineation ρ can be written as

$$l_2 = \infty\ 1_{a_1}\ 2_{a_2}\ 3_{a_3}\ 4_{a_4}$$

where a_1, a_2, a_3, a_4 are multiplicity of appearance of orbital numbers 1,2,3 and 4 of orbital block l_2.
Since $k = 48$ we have $a_1 + a_2 + a_3 + a_4 = 47$. Condition $|l_1 \cap l_2| = 12$ implies $a_1 = 11$. Hamming length of l_2, $H(l_2) = (|\rho| - 1) \cdot (\lambda - 1) = 506$, implies

$$a_1(a_1 - 1) + a_2(a_2 - 1) + a_3(a_3 - 1) + a_4(a_4 - 1) = 506$$

or

$$a_2(a_2 - 1) + a_3(a_3 - 1) + a_4(a_4 - 1) = 396 \text{ and } 0 \le a_i \le 19,\ i = 2, 3, 4.$$

In order to reduce isomorphic cases that may appear in the orbital structures at the last stage, without loss of generality, for the block l_2, we can use the reduction $a_2 \le a_3 \le a_4$.
We have proved that there exists only one orbital type for the block l_2, which satisfies above mentioned conditions

$$l_2 = \infty\ 1_{11}\ 2_{12}\ 3_{12}\ 4_{12}.$$

The third orbital block l_3, constructed by the collineation ρ, has the form $l_3 = 1_{b_1} 2_{b_2} 3_{b_3} 4_{b_4}$, where $b_1 + b_2 + b_3 + b_4 = 48$. Since $|l_1 \cap l_3| = 12$ we have $b_1 = 12$, and consequently $b_2 + b_3 + b_4 = 36$.
Since $H(l_3) = (|\rho| - 1) \cdot \lambda = 552$ we have

$$b_1(b_1 - 1) + b_2(b_2 - 1) + b_3(b_3 - 1) + b_4(b_4 - 1) = 552$$

or

$$b_2(b_2 - 1) + b_3(b_3 - 1) + b_4(b_4 - 1) = 420 \text{ and } 0 \le b_i \le 21,\ i = 2, 3, 4.$$

The Game Product $S(l_2, l_3) = |\rho| \cdot \lambda = 564$ implies $11b_1 + 12(b_2 + b_3 + b_4) = 564$. Obviously, among blocks l_3 are also blocks l_4 and l_5. Because that, we choose triples among candidates for the block l_3, such that every couple of them satisfies the condition of the Game Product. According to this fact we have found that, up to isomorphism and duality, there are exactly two orbital structures:

$$(1)\quad \begin{array}{llllll} l_1 = & \infty & 1_{47} & & & \\ l_2 = & \infty & 1_{11} & 2_{12} & 3_{12} & 4_{12} \\ l_3 = & & 1_{12} & 2_{8} & 3_{14} & 4_{14} \\ l_4 = & & 1_{12} & 2_{14} & 3_{8} & 4_{14} \\ l_5 = & & 1_{12} & 2_{12} & 3_{12} & 4_{8} \end{array} \qquad (2)\quad \begin{array}{llllll} l_1 = & \infty & 1_{47} & & & \\ l_2 = & \infty & 1_{11} & 2_{12} & 3_{12} & 4_{12} \\ l_3 = & & 1_{12} & 2_{16} & 3_{10} & 4_{10} \\ l_4 = & & 1_{12} & 2_{10} & 3_{16} & 4_{10} \\ l_5 = & & 1_{12} & 2_{10} & 3_{10} & 4_{16} \end{array}$$

In this way we have proved:

Theorem 2.2 *Up to isomorphism and duality there are exactly two orbital structures for a Symmetric Block Design $\mathcal{D}$ with parameters (189, 48, 12) admitting the collineation ρ of order 47, which on $\mathcal{D}$ operates with one fix point (block) while on the other points (blocks) of $\mathcal{D}$ operates transitively.*

Note 2.2 *The actual indexing of these two orbital structures in order to produce an example of design $\mathcal{D}$ is still an open problem.*

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