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Analytical Approximate Solution of Carleman's Equation by Using Maclaurin Series

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Abstract

In this paper we introduce a expansion method for solution of carleman's equation, in this method we expand the known function as a Maclaurin series and convert the solution into linear combination of some elements. we proved this linear combination is uniformly convergence to the analytic solution. If the known function be a polynomial then we have exact solution.

Mathematics Subject Classification: 45

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1 Introduction

In the theory of scattering of acoustic, electromagnetic, and earthquake waves by cylinders, infinite strips, and slits there arises a kind of weakly singular integral equations which is easily inverted as [8]

$$\int_{-a}^{a} \ln|x-t|\, g(t)dt = f(x) \qquad |x| < a,\ a \neq 2. \tag{1}$$

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Where $g(t)$ is unknown and $f(x)$ is known functions. Solution of (1) is[9],[2]

$$g(x) = \frac{1}{\pi^2}\int_{-a}^{a}\sqrt{\frac{a^2-t^2}{a^2-x^2}}\frac{f'(t)dt}{t-x} + \frac{1}{\pi^2\ln(\frac{a}{2})\sqrt{a^2-x^2}}\int_{-a}^{a}\frac{f(t)dt}{\sqrt{a^2-t^2}}. \qquad (2)$$

Integral equations with logarithmic kernels also arise in the boundary value problems for two-dimensional configurations [4, 5, 6, 7] and arised in plane elasticity crack problem, the dislocation distribution is taken as unknown function, and resultant force as the right hand term [1].

In this paper we consider $f(x) \in C^\infty[-a,a]$, and after expanding it in the form of Maclaurin series, substitute in equation (2) and split that integrals into some elements. In section 2 we evaluated these elements. Analytic and approximate solution in the form of series are obtained in section 3. Convergence of the mentioned series are discussed in section 4. Exact solution when the known function be a polynomial found in section 5.

2 Recurrence relations

Let $w_n = \int_{-a}^{a}\frac{t^n dt}{\sqrt{a^2-t^2}}$, then obviously

$$w_n = \begin{cases} \left(\frac{a}{2}\right)^{2k}\binom{2k}{k}\pi, & n = 2k, \\ 0, & n = 2k+1. \end{cases} \qquad (3)$$

Lemma 1 *Let* $I_n(x) = \int_{-a}^{a}\frac{t^n dt}{\sqrt{a^2-t^2}(x-t)}$ *then*

$$I_n(x) = -\pi\left[x^{n-1} + \sum_{j=1}^{m}\left(\frac{a}{2}\right)^{2j}\binom{2j}{j}x^{n-2j-1}\right]$$

Where $m = \frac{n}{2}-1$ *for* n *even, and* $m = \frac{n-1}{2}$ *for* n *odd. and recurrence relations*

$$I_{2k+1}(x) = xI_{2k}(x) - w_{2k},$$

$$I_{2k}(x) = xI_{2k-1}(x).$$

Proof: $I_n(x)$ can be written as

$$I_n(x) = \int_{-a}^{a}\frac{t^n - x^n}{\sqrt{a^2-t^2}(x-t)}dt + \int_{-a}^{a}\frac{x^n dt}{\sqrt{a^2-t^2}(x-t)}. \qquad (4)$$

It is known that (see [3]),

$$\int \frac{dt}{\sqrt{a^2-t^2}(x-t)} = \frac{1}{\sqrt{a^2-t^2}} \ln \left| \frac{t\sqrt{a^2-x^2} - x\sqrt{a^2-t^2}}{\sqrt{a^2-x^2}+\sqrt{a^2-t^2}} \right| + c \qquad (5)$$

From (5) it follows that the second integral is zero for $x \in (-a, a)$. Now using (3) and factorizing $(x-t)$ in the first integral of (4) we get

$$\begin{aligned} I_n(x) &= -\left(x^{n-1} \int_{-a}^{a} \frac{dt}{\sqrt{a^2-t^2}} + x^{n-2} \int_{-a}^{a} \frac{tdt}{\sqrt{a^2-t^2}} + \right. \\ &\quad \left. \cdots \; +x \int_{-a}^{a} \frac{t^{n-2}dt}{\sqrt{a^2-x^2}} + \int_{-a}^{a} \frac{t^{n-1}dt}{\sqrt{a^2-x^2}} \right) \\ &= -\left(x^{n-1}w_0 + x^{n-2}w_1 + \cdots + xw_{n-2} + w_{n-1} \right). \end{aligned} \qquad (6)$$

For finding the recurrence relations between $I_{2k+1}(x)$ and $I_{2k}(x)$ we consider two cases:
First let $n = 2k$, then

$$\begin{aligned} I_{2k}(x) &= -\pi \left(x^{2k-1} + a^2 \frac{2!}{4(1!)^2} x^{2k-3} + \cdots \right. \\ &\quad \left. + a^{2k-4} \frac{(2k-4)!}{4^{k-2}((k-2)!)^2} x^3 + a^{2k-2} \frac{(2k-2)!}{4^{k-1}((k-1)!)^2} x \right) \\ &= -\pi \left[x^{2k-1} + \sum_{j=1}^{k-1} \left(\frac{a}{2}\right)^{2j} \binom{2j}{j} x^{2k-2j-1} \right] \end{aligned}$$

Secondly, let $n = 2k+1$

$$\begin{aligned} I_{2k+1}(x) &= -\pi \left(x^{2k} + a^2 \frac{2!}{4(1!)^2} x^{2k-2} + \cdots \right. \\ &\quad \left. + a^{2k-2} \frac{(2k-2)!}{4^{k-1}((k-1)!)^2} x^2 + a^{2k} \frac{(2k)!}{4^k (k!)^2} \right) \\ &= -\pi \left[x^{2k} + \sum_{j=1}^{k} \left(\frac{a}{2}\right)^{2j} \binom{2j}{j} x^{2k-2j} \right], \ k = 1, 2, \cdots. \end{aligned}$$

Hence comparing these two equalities we obtain the recursive for I_n which are

$$I_{2k+1}(x) = xI_{2k}(x) - w_{2k},$$

$$I_{2k}(x) = xI_{2k-1}(x).$$

Combining of two above equations get the desirable result. With some manipulations we can get the results of Gakhov [2]

$$x^{n-1} + \sum_{k=1}^{m} \frac{1.3...(2k-1)}{2.4...2k} a^{2k} x^{n-2k-1}$$

■

3 Analytical approximate solution

Write (2) as

$$g(x) = \frac{1}{\pi^2\sqrt{a^2-x^2}} \left[A(x) + \frac{1}{\ln\frac{a}{2}} B \right] \tag{7}$$

where

$$A(x) = \int_{-a}^{a} \frac{(t^2-a^2)f'(t)dt}{\sqrt{a^2-t^2}(x-t)} \tag{8}$$

and

$$B = \int_{-a}^{a} \frac{f(t)dt}{\sqrt{a^2-t^2}} \cdot \tag{9}$$

We consider $f(x) \in C^{\infty}[-a,a]$ so it is possible to expand $f(x)$ as Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

3.1 Evaluation of $A(x)$

Write

$$(t^2-a^2)f'(t) = -f'(0)a^2 - f''(0)a^2 t + (f'(0) - \frac{f'''(0)}{2!}a^2)t^2 + \cdots$$

$$+(\frac{f^{(n-1)}(0)}{(n-2)!} - \frac{f^{(n+1)}}{n!}a^2)t^n + \cdots$$

Substituting in (8) yields

$$A(x) = -f^{'}(0)a^2I_0(x) - f^{''}(0)a^2I_1(x) + (f^{'}(0) - \frac{f^{'''}(0)}{2!}a^2)I_2(x) + \cdots$$

$$+(\frac{f^{(n-1)}(0)}{(n-2)!} - \frac{f^{(n+1)}}{n!}a^2)I_n(x) + \cdots \tag{10}$$

Where $I_n(x)$ is defined in (6).

3.2 Evaluation of B

Substituting Maclaurin series of $f(t)$ in (9) and using Lemma 1 gives

$$B = f(0)w_0 + f^{'}(0)w_1 + \frac{f^{''}(0)}{2!}w_2 + \cdots + \frac{f^{(2k)}(0)}{(2k)!}w_{2k} + \frac{f^{(2k+1)}(0)}{(2k+1)!}w_{2k+1} + \cdots$$

or

$$B = \pi\left[f(0) + \frac{f^{''}(0)}{4(1!)^2}a^2 + \frac{f^{(4)}(0)}{4^2(2!)^2}a^4 + \cdots + \frac{f^{(2k)}(0)}{4^k(k!)^2}a^{2k} + \cdots\right] \tag{11}$$

3.3 Approximation solution

The approximation solution is:

$$g_n(x) = \frac{1}{\pi^2\sqrt{a^2-x^2}}\left[A_n(x) + \frac{1}{\ln\frac{a}{2}}B_n\right] \tag{12}$$

where

$$A_n(x) = -f^{'}(0)a^2I_0(x) - f^{''}(0)a^2I_1(x) + (f^{'}(0) - \frac{f^{'''}(0)}{2!}a^2)I_2(x) + \cdots$$

$$+(\frac{f^{(n-1)}(0)}{(n-1)!} - \frac{f^{(n+1)}}{n!}a^2)I_n(x) \tag{13}$$

and

$$B_n = \pi\left[f(0) + \frac{f^{''}(0)}{4(1!)^2}a^2 + \frac{f^{(4)}(0)}{4^2(2!)^2}a^4 + \cdots + \frac{f^{(2k)}(0)}{4^k(k!)^2}a^{2k}\right] \tag{14}$$

where $n = 2k+1$ or $n = 2k$.

4 Error analysis

In the following convergence of $A_n(x)$ to $A(x)$ and B_n to B are proved respectively.

4.1 Error of $A_n(x)$

We are going to prove $A_n(x)$ is uniformly convergence to $A(x)$. From (10) and (13) it follows that

$$\begin{aligned}
&|A(x) - A_n(x)| \\
&= \left| \left(\frac{f^{(n)}(0)}{(n-1)!} - \frac{f^{(n+2)}(0)}{(n+1)!} a^2 \right) I_{n+1}(x) + \left(\frac{f^{(n+1)}(0)}{n!} - \frac{f^{(n+3)}(0)}{(n+2)!} a^2 \right) I_{n+2}(x) + \cdots \right| \\
&\leq \left| a^2 \left(\frac{f^{(n)}(0)}{(n-1)!} - \frac{f^{(n+2)}(0)}{(n+1)!} \right) I_{n+1}(x) + a^2 \left(\frac{f^{(n+1)}(0)}{n!} - \frac{f^{(n+3)}(0)}{(n+2)!} \right) I_{n+2}(x) + \cdots \right| \\
&\leq Ma^2 \left[\left(\frac{1}{(n-1)!} - \frac{1}{(n+1)!} \right) |I_{n+1}(x)| + \left(\frac{1}{n!} - \frac{1}{(n+2)!} \right) |I_{n+2}(x)| + \cdots \right] \\
&\leq Ma^2 \left[\left(\frac{1}{(n-1)!} - \frac{1}{(n+1)!} \right) (n+1) a^n + \left(\frac{1}{n!} - \frac{1}{(n+2)!} \right) (n+2) a^{n+1} + \cdots \right] \\
&\leq Ma^2 \left[\frac{1}{(n-1)!} (n+1) a^n + \frac{1}{n!} (n+2) a^{n+1} + \cdots \right] \\
&\leq Ma^2 \left[\left(\frac{1}{(n-2)!} + \frac{2}{(n-1)!} \right) a^n + \left(\frac{1}{(n-1)!} + \frac{2}{n!} \right) a^{n+1} + \cdots \right] \\
&\leq Ma^2 \left[\left(\frac{a^n}{(n-2)!} + \frac{a^{n+1}}{(n-1)!} + \cdots \right) + 2 \left(\frac{a^n}{(n-1)!} + \frac{a^{n+1}}{n!} + \cdots \right) \right] \\
&\leq Ma^3 \left[a \left(\frac{a^{n-2}}{(n-2)!} + \frac{a^{n-1}}{(n-1)!} + \cdots \right) + 2 \left(\frac{a^{n-1}}{(n-1)!} + \frac{a^n}{n!} + \cdots \right) \right] \\
&\leq Ma^3 \left[\frac{a^{n-1}}{(n-2)!} + (a+2) \left(\frac{a^{n-1}}{(n-1)!} + \frac{a^n}{n!} + \cdots \right) \right]
\end{aligned}$$

Since $\left|f^{(n)}(x)\right| \leq M$, and $|I_n(x)| \leq na^{n-1}$ for $n \in N$. After using Taylor theorem we have

$$|A(x) - A_n(x)| \leq Ma^3 \left[\frac{a^{n-1}}{(n-2)!} + \frac{a^n}{n!} e^{\xi_a} + \cdots \right], \ \xi_a \in (-a, a)$$

Since

$$\lim_{n \to \infty} \frac{a^n}{n!} = 0$$

so $A_n(x)$ is uniformly convergence to $A(x)$.

4.2 Error of B_n

We are going to prove B_n is convergence to B. From (11) and (14) it follows that

$$\begin{aligned} |B - B_n| &= \pi \left| \frac{f^{(2n+2)}(0)}{4^{n+1}(n+1)!} a^{2n+2} + \frac{f^{(2n+4)}(0)}{4^{n+2}(n+2)!} a^{2n+4} + \cdots \right| \\ &\leq \pi M \left[\frac{a^{2n+2}}{4^{n+1}(n+1)!} + \frac{a^{2n+4}}{4^{n+2}(n+2)!} + \cdots \right] \\ &\leq \pi M \left[\frac{(a^2)^{n+1}}{(n+1)!} + \frac{(a^2)^{n+2}}{(n+2)!} + \cdots \right] \\ &= \pi M \left[e^{a^2} - \left(1 + a^2 + \frac{(a^2)^2}{2!} \cdots + \frac{(a^2)^n}{n!} \right) \right] \end{aligned}$$

since

$$\lim_{n\to\infty} \left[e^{a^2} - \left(1 + a^2 + \frac{(a^2)^2}{2!} \cdots + \frac{(a^2)^n}{n!} \right) \right] = 0$$

B_n is convergence to B.

5 Exact solutions for polynomials

Let $f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_n x^n$, then

$$A_n(x) = -2\beta_2 a^2 I_1(x) + (\beta_1 - 3\beta_3 a^2) I_2(x) + \cdots + (\beta_{n-1} - (n+1)\beta_{n+1} a^2) I_n(x)$$

and

$$B_n = \beta_0 w_0 + \beta_2 w_2 + \cdots + \beta_{2k} w_{2k}$$

where $n = 2k + 1$ or $n = 2k$. Now we can obtain the solution from (12).

6 Conclusion

In this paper we use a polynomial as the known function of right hand side of Carleman's equation and find analytic solution for related integral equation also we found for class of infinite derivable function an approximate solution.

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Appendix

List of some $I_n(x)$ are as below:

$I_0(x) = 0.$

$I_1(x) = -\pi.$

$I_2(x) = -\pi x.$

$$I_3(x) = -\pi(x^2 + \tfrac{2!}{4(1!)^2}a^2).$$

$$I_4(x) = -\pi(x^3 + \tfrac{2!}{4(1!)^2}a^2 x).$$

$$I_5(x) = -\pi(x^4 + \tfrac{2!}{4(1!)^2}a^2x^2 + \tfrac{4!}{4^2(2!)^2}a^4).$$

$$I_6(x) = -\pi(x^5 + \tfrac{2!}{4(1!)^2}a^2x^3 + \tfrac{4!}{4^2(2!)^2}a^4x).$$

$$I_7\left(x\right) = -\pi\left(x^6 + \tfrac{2!}{4(1!)^2}a^2x^4 + \tfrac{4!}{4^2(2!)^2}a^4x^2 + \tfrac{6!}{4^3(3!)^2}a^6\right).$$

$$I_8\left(x\right) = -\pi\left(x^7 + \tfrac{2!}{4(1!)^2}a^2x^5 + \tfrac{4!}{4^2(2!)^2}a^4x^3 + \tfrac{6!}{4^3(3!)^2}a^6x\right).$$

$$I_9\left(x\right) = -\pi\left(x^8 + \tfrac{2!}{4(1!)^2}a^2x^6 + \tfrac{4!}{4^2(2!)^2}a^4x^4 + \tfrac{6!}{4^3(3!)^2}a^6x^2 + \tfrac{8!}{4^4(4!)^2}a^8\right).$$

$$I_{10}\left(x\right) = -\pi\left(x^9 + \tfrac{2!}{4(1!)^2}a^2x^7 + \tfrac{4!}{4^2(2!)^2}a^4x^5 + \tfrac{6!}{4^3(3!)^2}a^6x^3 + \tfrac{8!}{4^4(4!)^2}a^8x\right).$$

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