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Strong Convergence Theorem by Hybrid Iterative Scheme for Generalized Equilibrium Problems and Fixed Point Problems of Strictly Pseudo-Contraction Mappings

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Abstract

The purpose of this work is to introduce a hybrid iterative scheme for finding a common element of the set of a generalized equilibrium problem, the set of solutions to a variational inequality and the set of fixed points of a strict pseudo-contraction mappings in a real Hilbert space. The results obtained in this paper extend and improve the result of Cho, Qin and Kang [Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems, Nonlinear Anal. doi:10.1016/j.na.2009.02.106], and many authors.

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping S of C into itself is nonexpansive if $\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C$. The set of fixed points of S is denoted by $F(S)$. Let F be

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a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the real numbers. The equilibrium problem for $F : C \times C \to \mathbf{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all} \quad y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1). In 1997, Combettes and Hirstoaga [6] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $VI(C, A)$, is to find $x^* \in C$ such that $\langle Ax^*, v - x^* \rangle \geq 0$ for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [15] and the references therein. Let $B : C \to H$ be a nonlinear mapping. Then, we consider the following generalized equilibrium problem(GEP): Find $z \in C$ such that

$$F(z, y) + \langle Bz, y - z \rangle \geq 0, \forall y \in C \tag{2}$$

The set of such $z \in C$ is denoted by EP, i.e.,

$EP = \{z \in C : F(z, y) + \langle Bz, y - z \rangle \geq 0, \forall y \in C\}$.

In the case of $B \equiv 0$, EP is denoted by $EP(F)$. In the case of $F \equiv 0$, EP is also denoted by $VI(C, A)$. A mapping A of C into H is called α-inverse-strongly monotone [2] if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2$$

for all $u, v \in C$. Recently, Takahashi and Toyoda [11] and Yao et al. [16] introduced an iterative method for finding an element of $VI(C, A) \cap F(S)$, where $A : C \to H$ is an α-inverse-strongly monotone mapping. Let A be a strongly positive bounded linear operator on H: that is, there exists a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2 \quad \text{for all} \quad x \in H. \tag{3}$$

A mapping $S : C \to C$ is called a *k-strict pseudo-contraction mapping* if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \tag{4}$$

for all $x, y \in C$. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty. It is well-known that S is nonexpansive if and only if S is 0-strictly pseudo-contractive. The mapping S is also said to be pseudo-contractive if $k = 1$ and S is said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0, 1)$ such that $S - \lambda I$ is pseudo-contractive. Clearly, the class of k-strictly pseudo-contractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of k-strictly pseudo-contractive mappings.

In 1967, Browder and Petryshyn [2] established the first convergence result for k-strict pseudo-contraction in real Hilbert spaces. They proved weak and strong convergence theorem by using iteration with a constant control sequence $\{\alpha_n\} = \alpha$ for all n. Many authors have appeared in the literature on the existence of solution equilibrium, see also, for example [1, 5, 8, 12] and references therein. To find an element of $EP(F) \cap F(S)$, Takahashi and Takahashi [12] introduced the an iterative scheme for nonexpansive mappings by the hybrid method in a Hilbert space.

Recently, in 2008, Takahashi and Takahashi [10] introduced a hybrid iterative method for finding a common element of EP and $F(S)$. They defined $\{x_n\}$ in the following way:

$$\begin{cases} u_n \in C, \quad \text{such that} \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S(a_n u + (1 - a_n u_n)), \quad \forall n \in \mathbf{N}. \end{cases} \tag{5}$$

where B be an β-inverse strongly monotone mapping of C into H with positive real number α, and proved strong convergence theorems in the framework of a Hilbert space, under some suitable conditions on parameters $\{a_n\}, \{\beta_n\}$ and $\{\lambda_n\}$.

Very recently, Cho, et al. [4], Ceng et al. [5], Liu [7] and Peng et al. [9] established an iterative scheme for finding a common element of the set of solution of an equilibrium problem (1), generalized equilibrium problem (2) and the set of fixed point of a k-strict pseudo-contraction mapping in the setting of real Hilbert space. They also studied some weak and strong convergence theorem for k-strict pseudo-contraction mappings of the sequence generated by their algorithm.

In 2009, Cho, Qin and Kang [3] introduce the hybrid methods for finding a common element of $F(S) \cap VI(C, A) \cap EP$. Let S be a k-strict pseudo-contraction mapping and defined ${}_K x = kx(1 - k)Sx$ for all $x \in C$. They defined $\{x_n\}$ in the following way:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n) \\ z_n = \alpha_n x_n + (1 - \alpha_n)S_k y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \ n \geq 1 \end{cases}$$

A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. They proved strong convergence theorems in the framework of a Hilbert space, under some suitable conditions on parameters $\{a_n\}, \{r_n\}$ and $\{\lambda_n\}$.

In this paper, we extend and improve the result of Cho, Qin and Kang [3]. Then, we obtain the strong convergence theorem for the sequences generated by these processes. Furthermore, using the theorem we also obtain strong convergence theorems for finding elements of fixed points, equilibrium problems and the set of solutions to a variational inequality, respectively.

2 Preliminary

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denote by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfied

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{6}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following propertied: $P_C x \in C$ and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{7}$$

for all $x \in H$, $y \in C$. The following is the property in Hilbert spaces: for any $x, y \in H$, we have

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$
(ii) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$
(iii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$
(iv) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \ \forall t \in [0, 1]$.

Remark 2.1 *We note that if A is a α-inverse-strongly monotone, for all $u, v \in C$ and $\lambda_n > 0$,*

$$\begin{aligned}\|(I - \lambda_n A)u - (I - \lambda_n A)v\|^2 &= \|(u - v) - \lambda_n(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda_n\langle u - v, Au - Av\rangle \\ &\quad + \lambda_n^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au - Av\|^2. \end{aligned} \tag{8}$$

So, if $\lambda_n \leq 2\alpha$, then $I - \lambda_n A$ is a nonexpansive mapping from C to H.

Lemma 2.2 *[17] Let $T : K \to H$ be a k-strictly pseudo-contraction. Defined $D : K \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for each $x \in K$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbf{R}$, let us assume that F satisfies the following condition:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y \in C$, $\lim_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.3 *[1] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that $F(z, y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0$ for all $y \in C$.*

Lemma 2.4 [1, 6, 10] *Assume that $F : C \times C \to \mathbf{R}$ satisfies (A1)-(A4), and let $r > 0$ and $x \in H$. Then,there exists unique $z \in C$ such that $F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$ for all $y \in C$. Moreover, let T_r be a mapping of H into C defined by $T_r(x) = z$ for all $x \in H$. Then, the following hold:*

1.T_r is single- valued;

2.T_r is firmly nonexpansive, i.e., $\|T_rx - T_ry\|^2 \leq \langle T_rx - T_ry, x - y\rangle$, for any $x, y \in H$;

3.$F(T_r) = EP(F)$;

4.$EP(F)$ is closed and convex;

5.$\|T_sx - T_tx\|^2 \leq \frac{s-t}{s}\langle T_sx - T_tx, T_sx - x\rangle$, for all $s, t > 0$ and $x \in H$.

Lemma 2.5 *(see [13, 14]) Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property,*

$$a_{n+1} \leq (1 - \gamma_n)a_n + b_n, \ n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$, and $\{b_n\}$ is a sequence in $\mathbf{R}$ such that:

i) $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;

ii) $\limsup_{n\to\infty} \frac{b_n}{\gamma_n} \leq 0$ or $\Sigma_{n=1}^{\infty}|b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main Results

In this section, we prove a strong convergence theorem of the hybrid method for strictly pseudo-contractive mappings in a real Hilbert space.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a k-strictly pseudo-contractive for some $0 \leq k < 1$. Defined a mapping*

$S_k : C \to C$ by $S_k x = kx + (1-k)Sx$ for all $x \in C$. Assume that $F := F(S) \cap VI(C,A) \cap EP \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by

$$\begin{cases} x_1 = C, \\ C_1 = C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n) \\ w_n = \beta_n S_k y_n + (1-\beta_n) y_n, \\ z_n = \alpha_n x_n + (1-\alpha_n) S_k w_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \ n \geq 1 \end{cases} \tag{9}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$ and $\{r_n\} \subset (0,\infty)$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1], \{\lambda_n\} \subset (0, 2\alpha)$ and $\{r_n\} \subset (0, 2\beta)$ satisfy the following conditions:
(i) $k \leq \alpha_n, \beta_n \leq a < 1$,
(ii) $0 \leq b \leq \lambda_n \leq c < 2\alpha$ and $0 \leq d \leq r_n \leq e < 2\beta$,for some $a, b, c, d, e \in \mathbf{R}$.
Then $\{x_n\}$ converge strongly to z, where $z = P_F x_1$.

Proof. Let $p \in F$ since $0 \leq r_n < 2\beta$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n B x_n) - T_{r_n}(p - r_n B p)\|^2 \\ &\leq \|(x_n - r_n B x_n) - (p - r_n B p)\|^2 \\ &\leq \|(x_n - p) - r_n(Bx_n - Bp)\|^2 \\ &\leq \|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle + r_n^2\|Bp - Bx_n\|^2 \\ &\leq \|x_n - p\|^2 - 2r_n\beta\|Bx_n - Bp\|^2 + r_n^2\|Bp - Bx_n\|^2 \qquad (10) \\ &\leq \|x_n - p\|^2. \qquad (11) \end{aligned}$$

First we show that $F \subset C_n$ for all $n \in \mathbf{N}$, we can prove by induction. It is obvious that $F \subset C_1$. Let $p \in F$, we known that $I - \lambda_n A$ is nonexpansive, for all $n \in \mathbf{N}$ and from $p \in VI(C,A)$ we get $p = P_C(p - \lambda_n A p)$. It follows that

$$\begin{aligned} \|y_n - p\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(p - \lambda_n A p)\|^2 \\ &\leq \|(I - \lambda_n A)u_n - (I - \lambda_n A)p\|^2 \\ &\leq \|u_n - p\|^2. \qquad (12) \end{aligned}$$

Consider,

$$\begin{aligned} \|w_n - p\| &= \|\beta_n(S_k y_n - p) + (1-\beta_n)(y_n - p)\| \\ &\leq \beta_n\|S_k y_n - p\| + (1-\beta_n)\|y_n - p\| \\ &\leq \beta_n\|y_n - p\| + (1-\beta_n)\|y_n - p\| \\ &= \|y_n - p\| \\ &= \|u_n - p\| \\ &= \|x_n - p\|. \qquad (13) \end{aligned}$$

Thus, we have

$$
\begin{aligned}
\|z_n - p\| &= \|\alpha_n x_n + (1-\alpha_n) S_k w_n - p\| \\
&= \|\alpha_n (x_n - p) + (1-\alpha_n)(S_k w_n - p)\| \\
&\le \alpha_n \|x_n - p\| + (1-\alpha_n)\|S_k w_n - p\| \qquad (14) \\
&\le \alpha_n \|x_n - p\| + (1-\alpha_n)\|w_n - p\| \\
&\le \alpha_n \|x_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&= \|x_n - p\|. \qquad (15)
\end{aligned}
$$

So, we have $p \in C_{n+1}$ and hence $F \subset C_n$, for all $n \in \mathbf{N}$.

Next, we show that C_n is closed and convex for all $n \in \mathbf{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbf{N}$. Let $c_j \in C_{m+1} \subset C_m$ with $c_j \to z$. Since C_m is closed, $z \in C_m$ and $\|z_m - c_j\| \le \|c_j - x_m\|$. Then

$$
\begin{aligned}
\|z_m - z\| &= \|z_m - c_j + c_j - z\| \\
&\le \|z_m - c_j\| + \|c_j - z\| \qquad (16)
\end{aligned}
$$

Taking $j \to \infty$, we have $\|z_m - z\| \le \|z - x_m\|$. Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha) y$ where $\alpha \in [0,1]$. Since C_m is convex, $z \in C_m$ and $\|z_m - x\| \le \|x - x_m\|, \|z_m - y\| \le \|y - x_m\|$, we have

$$
\begin{aligned}
\|z_m - z\|^2 &= \|z_m - (\alpha x + (1-\alpha) y)\|^2 \\
&= \|\alpha (z_m - x) + (1-\alpha)(z_m - y)\|^2 \\
&= \alpha \|z_m - x\|^2 + (1-\alpha)\|z_m - y\|^2 - \alpha(1-\alpha)\|(z_m - x) - (z_m - y)\|^2 \\
&\le \alpha \|z_m - x\|^2 + (1-\alpha)\|z_m - y\|^2 - \alpha(1-\alpha)\|y - x\|^2 \\
&\le \alpha \|x_m - x\|^2 + (1-\alpha)\|x_m - y\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\
&= \|x_m - (\alpha x + (1-\alpha) y)\|^2 \\
&= \|x_m - z\|^2. \qquad (17)
\end{aligned}
$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbf{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n} x_1$, we have $\langle x_1 - x_n, x_n - y \rangle \ge 0$, for all $y \in C_n$. Since $F \subset C_n$, we obtain

$$
\langle x_1 - x_n, x_n - u \rangle \ge 0 \text{ for all } u \in F \text{ and } n \in \mathbf{N}. \qquad (18)
$$

So, for $u \in F$, we get

$$
\begin{aligned}
0 \le \langle x_1 - x_n, x_n - u \rangle &= \langle x_1 - x_n, x_n - x_1 + x_1 - u \rangle \\
&= -\langle x_n - x_1, x_n - x_1 \rangle + \langle x_1 - x_n, x_1 - u \rangle \\
&\le -\|x_n - x_1\|^2 + \|x_1 - x_n\| \|x_1 - u\|.
\end{aligned}
$$

This implies that $\|x_1 - x_n\|^2 \le \|x_1 - x_n\| \|x_1 - u\|$, hence

$$
\|x_1 - x_n\| \le \|x_1 - u\| \text{ for all } u \in F \text{ and } n \in \mathbf{N}. \qquad (19)
$$

From $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have

$$\langle x_1 - x_n, x_n - x_{n+1}\rangle \geq 0 \ \text{ for all } n \in \mathbf{N}. \tag{20}$$

So, for $x_{n+1} \in C_n$, we also have, for $n \in \mathbf{N}$

$$\begin{aligned} 0 \leq \langle x_1 - x_n, x_n - x_{n+1}\rangle &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\ &= -\langle x_n - x_1, x_n - x_1\rangle + \langle x_1 - x_n, x_1 - x_{n+1}\rangle \\ &\leq -\|x_n - x_1\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|. \end{aligned}$$

This implies that $\|x_1 - x_n\|^2 \leq \|x_1 - x_n\|\|x_1 - x_{n+1}\|$ and we get

$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\| \ \text{ for all } n \in \mathbf{N}. \tag{21}$$

From (19), we have $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - x_1\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (20), we note that

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_1) + (x_1 - x_{n+1})\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\langle x_1 - x_n, x_1 - x_n\rangle - 2\langle x_1 - x_n, x_n - x_{n+1}\rangle \\ &\quad + \|x_1 - x_{n+1}\|^2 \\ &\leq \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + \|x_1 - x_{n+1}\|^2 \\ &= -\|x_n - x_1\|^2 + \|x_1 - x_{n+1}\|^2. \end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, we obtain

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{22}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ imply that

$$\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \to 0 \text{ as } n \to \infty. \tag{23}$$

Further, we get $\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|$.
From (22) and (23), we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{24}$$

Next, we show that $\lim_{n\to\infty}\|x_n - u_n\| = 0$. For $p \in \Theta$. From (13), (10) and by (ii), we have

$$\begin{aligned} \|z_n - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(S_k w_n - p)\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|S_k w_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - S_k w_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|w_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - S_k w_n\|^2 \qquad (25) \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[\|x_n - p\|^2 - 2r_n\beta\|Bx_n - Bp\|^2 \\ &\quad + r_n^2\|Bp - Bx_n\|^2] \qquad (26) \\ &= \|x_n - p\|^2 + d(e - 2\beta)\|Bx_n - Bp\|^2, \end{aligned}$$

and hence

$$\begin{aligned} d(2\beta - e)\|Bx_n - Bp\|^2 &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\ &= \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|). \end{aligned}$$

From (24), we have $\lim_{n\to\infty} \|Bx_n - Bp\| = 0$. From remark 2.1 that for $\lambda_n \le 2\beta$ then $I - r_nB$ is nonexpansive, for all $n \in \mathbf{N}$, T_{r_n} is firmly nonexpansive and by using Lemma 2.4, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(p - r_nBp)\|^2 \\ &\le \langle (x_n - r_nBx_n) - (p - r_nBp), u_n - p\rangle \\ &= \frac{1}{2}(\|(x_n - r_nBx_n) - (p - r_nBp)\|^2 + \|u_n - p\|^2 \\ &\quad - \|(x_n - r_nBx_n) - (p - r_nBp) - (u_n - p)\|^2) \\ &\le \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(Bx_n - Bp)\|^2) \\ &= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2). \end{aligned}$$

Thus, we obtain

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2. \tag{27}$$

From (27), we have

$$\begin{aligned} \|z_n - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(S_kw_n - p)\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|S_kw_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - S_kw_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|w_n - p\|^2 \qquad (28) \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[\|x_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2] \qquad (29) \\ &\le \|x_n - p\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Bx_n - Bp\|, \end{aligned}$$

it follows that

$$\begin{aligned} (1 - a)\|x_n - u_n\|^2 &\le (1 - \alpha_n)\|x_n - u_n\|^2 \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 + 2r_n\|x_n - u_n\|\|Bx_n - Bp\| \\ &\le \|x_n - z_n\|(\|x_n - p\| - \|z_n - p\|) \\ &\quad + 2r_n\|x_n - u_n\|\|Bx_n - Bp\|. \end{aligned}$$

Using (24) and $\|Bx_n - Bp\| \to 0$, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{30}$$

Consider,

$$\begin{aligned}\|w_n - p\|^2 &= \|\beta_n(S_k y_n - p) + (1-\beta_n)(y_n - p)\|^2\\ &= \beta_n\|S_k y_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2\\ &\le \beta_n\|y_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2\\ &= \|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2, \qquad (31)\end{aligned}$$

From (28) and (31) we also have

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2\\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2] \qquad (32)\\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2\\ &\le \|x_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}(1-a)k(1-a)\|y_n - S_k y_n\|^2 &\le (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2\\ &\le \|x_n - p\|^2 - \|z_n - p\|^2\\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).\end{aligned}$$

From (24), we have

$$\lim_{n\to\infty}\|S_k y_n - y_n\| = 0. \qquad (33)$$

Next, we show that $\lim_{n\to\infty}\|u_n - y_n\| = 0$. Consider

$$\begin{aligned}\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(p - \lambda_n Ap)\|^2\\ &\le \|(u_n - \lambda_n Au_n) - (p - \lambda_n Ap)\|^2\\ &= \|(u_n - p) - \lambda_n(Au_n - Ap)\|^2\\ &= \|u_n - p\|^2 - \lambda_n\langle u_n - p, Au_n - Ap\rangle + \lambda_n^2\|Au_n - Ap\|^2\\ &\le \|x_n - p\|^2 - 2\lambda_n\alpha\|Au_n - Ap\|^2 + \lambda_n^2\|Au_n - Ap\|^2\\ &= \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Ap\|^2\\ &\le \|x_n - p\|^2 + b(c - 2\alpha)\|Au_n - Ap\|^2.\end{aligned}$$

From (32) and (ii), we have

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2\\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 + b(c-2\alpha)\|Au_n - Ap\|^2]\\ &\le \|x_n - p\|^2 + (1-\alpha_n)b(c-2\alpha)\|Au_n - Ap\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}(1-a)b(2\alpha - c)\|Au_n - Ap\|^2 &\le (1-\alpha_n)b(2\alpha - c)\|Au_n - Ap\|^2\\ &\le \|x_n - p\|^2 - \|z_n - p\|^2\\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).\end{aligned}$$

From (24), that

$$\lim_{n\to\infty} \|Au_n - Ap\| = 0. \tag{34}$$

From (6), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(p - \lambda_n Ap)\|^2 \\
&\le \langle (u_n - \lambda_n Au_n) - (p - \lambda_n Ap), y_n - p\rangle \\
&= \frac{1}{2}\{\|(u_n - \lambda_n Au_n) - (p - \lambda_n Ap)\|^2 + \|y_n - p\|^2 \\
&\quad -\|(u_n - \lambda_n Au_n) - (p - \lambda_n Ap) - (y_n - p)\|^2\} \\
&\le \frac{1}{2}\{\|u_n - p\|^2 + \|y_n - p\|^2 - \|(u_n - y_n) - \lambda_n(Au_n - Ap)\|^2\} \\
&= \frac{1}{2}\{\|u_n - p\|^2 + \|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle \\
&\quad -\lambda_n^2\|Au_n - Ap\|^2\},
\end{aligned}$$

so, we obtain

$$\|y_n - p\|^2 \le \|u_n - p\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle - \lambda_n^2\|Au_n - Ap\|^2. \tag{35}$$

From (12), (32), (35) and (i) we have

$$\begin{aligned}
\|z_n - p\|^2 &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|u_n - p\|^2 - \|u_n - y_n\|^2 \\
&\quad +2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle - \lambda_n^2\|Au_n - Ap\|^2] \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)\|u_n - y_n\|^2 \\
&\quad +(1-\alpha_n)2\lambda_n\|u_n - p\|\|Au_n - Ap\| \\
&\le \|x_n - p\|^2 - (1-\alpha_n)\|u_n - y_n\|^2 + 2\lambda_n\|u_n - y_n\|\|Au_n - Ap\|,
\end{aligned}$$

it follows that

$$\begin{aligned}
(1-a)\|u_n - y_n\|^2 &\le (1-\alpha_n)\|u_n - y_n\|^2 \\
&\le \|x_n - p\|^2 - \|z_n - p\|^2 + 2\lambda_n\|u_n - y_n\|\|Au_n - Ap\| \\
&\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2\lambda_n\|u_n - p\|\|Au_n - Ap\|,
\end{aligned}$$

From (i), (24) and (34), we obtain

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{36}$$

Next, we show that $\lim_{n\to\infty}\|S_k u_n - u_n\| = 0$, consider

$$\begin{aligned}
\|S_k u_n - u_n\| &\le \|S_k u_n - S_k y_n\| + \|S_k y_n - y_n\| + \|y_n - u_n\| \\
&\le 2\|y_n - u_n\| + \|S_k y_n - y_n\|.
\end{aligned}$$

From (36) and (33) we obtain that

$$\lim_{n\to\infty} \|S_k u_n - u_n\| = 0. \tag{37}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ such that $u_{n_{i_j}} \rightharpoonup w$. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. Since C is closed and convex, $w \in C$. Next, we show that $w \in F$. First, we show that $w \in VI(C, A)$. Define,

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{38}$$

Then, T is maximal monotone. Let $(v, u) \in G(T)$. Since $u - Av \in N_C v$ and $y_n \in C$, we have $\langle v - y_n, u - Av\rangle \geq 0$. On the other hand, from $y_n = P_C(u_n - \lambda_n A u_n)$, we have $\langle v - y_n, y_n - (u_n - \lambda_n A u_n)\rangle \geq 0$, that is,

$$\left\langle v - y_n, \frac{y_n - u_n}{\lambda_n} + Au_n \right\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - y_{n_i}, u\rangle &\geq \langle v - y_{n_i}, Av\rangle \\
&\geq \langle v - y_{n_i}, Av\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i} \right\rangle \\
&= \left\langle v - y_{n_i}, Av - Au_{n_i} - \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - y_{n_i}, Av - Ay_{n_i}\rangle + \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle,
\end{aligned}$$

Since $\lim_{n\to\infty} \|y_n - u_n\| = 0$ and A is Lipschitz continuous, we obtain

$$\langle v - w, u\rangle \geq 0. \tag{39}$$

Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, A)$. Next, we show that $w \in EP$. It follows by (9) and (A2) that

$$\langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}) \tag{40}$$

Put $y_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. Since $y \in C$ and $w \in C$, we have $y_t \in C$. So, from (30), we have

$\langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, By_t\rangle = 0 \geq -\langle y_t - u_{n_i}, Bx_{n_i}\rangle - \langle y - u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i})$

and hence

$\langle y_t - u_{n_i}, By_t\rangle \geq \langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, Bx_{n_i}\rangle - \langle y - u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i})$

$= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle - \langle y - u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i})$.

Since $\|u_{n_i} - x_{n_i}\| \to 0$, it follows that $\|Bu_{n_i} - Bx_{n_i}\| \to 0$. Further, from monotonicity of B, we get $\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \geq 0$. So, from (A4), we have

$$\langle y_t - w, By_t\rangle \geq F(y_t, w), \tag{41}$$

as $i \to \infty$. From (A1), (A4) and (41), we have

$$\begin{aligned} 0 = F(y_t, y_t) &\leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y) + (1-t)\langle y_t - w, By_t\rangle \\ &\leq tF(y_t, y) + (1-t)t\langle y - w, By_t\rangle \end{aligned}$$

and hence $0 \leq F(y_t, y) + (1-t)\langle y - w, By_t\rangle$. Letting $t \to 0$, we have for each $y \in C$, $0 \leq F(w, y) + \langle y - w, Bw\rangle$. This implies that $w \in EP$. Next, we show that $w \in F(S)$. From Lemma 2.2, we have $F(S_k) = F(S)$, we may assume that $w \neq S_k w$, by Opial's condition, we have

$$\begin{aligned} \liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - S_k w\| = \liminf_{i\to\infty} \|(u_{n_i} - S_k u_{n_i}) + (S_k u_{n_i} - S_k w)\| \\ &= \liminf_{i\to\infty} \|S_k u_{n_i} - S_k w\| \leq \liminf_{i\to\infty} \|u_{n_i} - w\|. \end{aligned}$$

This is a contradiction. So, we have $w \in F(S_k) = F(S)$. Therefore $w \in F$.

Finally, we show that $x_n \to z$, where $z = P_F x_1$. Since $x_n = P_{C_n} x_1$ and $z \in F \subset C_n$, we have $\|x_n - x_1\| \leq \|z - x_1\|$. It follows from $z' = P_F x_1$ and the lower semicontinuity of the norm that

$$\|z' - x_1\| \leq \|z - x_1\| \leq \liminf_{i\to\infty} \|x_{n_i} - x_1\| \leq \limsup_{i\to\infty} \|x_{n_i} - x_1\| \leq \|z' - x_1\|. \tag{42}$$

Thus, we obtain that $\lim_{k\to\infty} \|x_{n_i} - x_1\| = \|z - x_1\| = \|z' - x_1\|$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z, where $z = P_F x_1$. ◇

Theorem 3.2 *[3] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a k-strictly pseudo-contractive self mapping for some $0 \leq k < 1$. Defined a mapping $S_k : C \to C$ by $S_k x = kx + (1-k)Sx$ for all $x \in C$. Assume* *that* $F := F(S) \cap VI(C, A) \cap EP \neq \emptyset$. *Let the sequences* $\{x_n\}$ *and* $\{u_n\}$ *be generated by* $C_1 = C \subset H$, $x_1 = P_C x_0$*;*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n) \\ z_n = \alpha_n x_n + (1 - \alpha_n) S_k y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0 \end{cases} \tag{43}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$ *and* $\{r_n\} \subset (0, \infty)$. *Assume that the control sequences* $\{\alpha_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, 2\alpha)$ *and* $\{r_n\} \subset (0, 2\beta)$ *satisfy the following conditions:* (i) $k \leq \alpha_n \leq a < 1$, (ii) $0 \leq b \leq \lambda_n \leq c < 2\alpha$ *and* $0 \leq d \leq r_n \leq e < 2\beta$, *for some* $a, b, c, d, e \in \mathbf{R}$.
Then $\{x_n\}$ *converge strongly to* z, *where* $z = P_F x_0$.

Proof. If $\beta_n = 0$ for all $n \in \mathbf{N}$, by Thm 3.1, we obtain the desired result. $\diamond$

Corollary 3.3 [4, Theorem 3.1] *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $F : C \times C \to \mathbf{R}$ *be a bifunction satisfying (A1)-(A4). Let* B *be an* β*-inverse-strongly monotone mapping of* C *into* H. *Let* $T : C \to C$ *be a* k*-strictly pseudo-contractive self mapping for some* $0 \leq k < 1$ *such that* $\Theta := F(T) \cap VI(C, A) \cap EP \neq \emptyset$. *Let the sequences* $\{x_n\}$ *and* $\{u_n\}$ *be generated by* $C_1 = C \subset H$, $x_1 = P_C x_0$*;*

$$\begin{cases} u_n \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ z_n = \alpha_n x_n + (1 - \alpha_n) S u_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0 \end{cases} \tag{44}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$, $S = kx + (1 - k)T$ *for all* $x \in C$ *and* $\{r_n\} \subset (0, \infty)$. *Assume that the control sequences* $\{\alpha_n\} \subset [0, 1]$ *and* $\{r_n\} \subset (0, 2\beta)$ *satisfy the following conditions:*

(i) $0 \leq k < \alpha_n \leq a < 1$,

(ii) $0 \leq d \leq r_n \leq e < 2\beta$, *for some* $a, d, e \in \mathbf{R}$.

Then $\{x_n\}$ *converge strongly to* z, *where* $z = P_\Theta x_0$.

Proof. Put $A \equiv 0$, $\beta_n = 0$ for all $n \in \mathbf{N}$, and from Lemma 2.2, we have $F(S) = F(T)$ and Theorem 3.1, we obtain the desired result. $\diamond$

4 Applications

In this section, we obtain some strong convergence theorems by applying $F \equiv 0$ and $\beta_n \equiv 0$ for all $n \in \mathbf{N}$ in Theorem 3.1.

Theorem 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $F := F(S) \cap VI(C, A) \cap EP \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by*

$$\begin{cases} x_1 = C, \\ C_1 = C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n) \\ z_n = \alpha_n x_n + (1 - \alpha_n) S y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \ n \geq 1 \end{cases} \tag{45}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$ and $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\} \subset [0, 1], \{\lambda_n\} \subset (0, 2\alpha)$ and $\{r_n\} \subset (0, 2\beta)$ satisfy the following conditions:
(i) $k \leq \alpha_n \leq a < 1$,
(ii) $0 \leq b \leq \lambda_n \leq c < 2\alpha$ and $0 \leq d \leq r_n \leq e < 2\beta$, for some $a, b, c, d, e \in \mathbf{R}$.
Then $\{x_n\}$ converge strongly to z, where $z = P_F x_1$.

Theorem 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a k-strictly pseudo-contractive for some $0 \leq k < 1$. Defined a mapping $S_k : C \to C$ by $S_k x = kx + (1 - k)Sx$ for all $x \in C$. Assume that $F := F(S) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $C_1 = C \subset H$, $x_1 = P_C x_0$ and $u_n \in C$;*

$$\begin{cases} \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n) \\ z_n = \alpha_n x_n + (1 - \alpha_n) S_k y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0 \end{cases} \tag{46}$$

where $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\} \subset [0, 1], \{\lambda_n\} \subset (0, 2\alpha)$ and $\{r_n\} \subset (0, 2\beta)$ satisfy the following conditions:

(i) $k \leq \alpha_n \leq a < 1$,
(ii) $0 \leq b \leq \lambda_n \leq c < 2\alpha$ *and* $0 \leq d \leq r_n \leq e < 2\beta$, *for some* $a, b, c, d, e \in \mathbf{R}$.
Then $\{x_n\}$ *converge strongly to* z, *where* $z = P_F x_0$.

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