A Note on Extension of p.q. Baer Rings

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Abstract

Let $R$ be a ring. Let $\alpha$ be an endomorphism of $R$. In this paper it is shown that let $R$ be $\alpha$-Armendariz ring of power series wise then the following are equivalent

(i) $R[[x,\alpha]]$ is right p.q. Baer ring.

(ii) $R$ is right p.q. Baer ring and any countable family of idempotents of $R$ has generalized join in $I(R)$.

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1 Introduction

Through this paper $R$ denote associated ring with unity and $I(R)$ be a set of all idempotents of $R$. Recall that $R$ is (quasi)-Baer ring if the right annihilator of every non-empty subset (ever right ideal) of $R$ is generated by an idempotent. In [16], Kaplansky introduced Baer rings to abstract various properties of AW*-Algebras and von-Neumann algebras. In [8], Clark defined quasi-Baer rings and used them to characterize when a finite dimensional algebra with unity over a algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Another generalization of Baer rings is p.p.-ring. A ring $R$ is called a right p.p. (resp. left p.p.) ring if the right (resp., left) annihilator of an element of $R$ is generated by an idempotent. As a another generalization of quasi-Baer rings, in [5], G.F. Birkenmeier J.Y. Kim, and J.K. Park, introduced the concept of Principally quasi-Baer rings. A ring $R$ is called p.q. Baer ring if the right annihilator of principal right ideal of $R$ is generated by an idempotent.
Similarly, left p.q. Baer rings can be defined. A ring $R$ is called p.q. Baer if it is left and right p.q. Baer ring. The class of p.q. Baer rings include all biregular rings, all quasi-Baer rings, and all abelian p.p.-rings.

We write $R[x], R[[x]]$ for the polynomial ring, and the power series ring, respectively. Consider

$$R[x, \alpha] = \left\{ \sum_{i=0}^{p} a_i x^i \mid p \geq 0, a_i \in R \right\}$$

$$R[[x, \alpha]] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in R \right\}$$

Each of these is an abelian group under an obvious addition operation. Moreover, $R[x, \alpha]$ becomes a ring under the following product operation: for

$$f(x) = \sum_{i=0}^{p} a_i x^i,$$

$$g(x) = \sum_{j=0}^{q} b_j x^j \in R[x, \alpha]$$

$$f(x)g(x) = \sum_{k=0}^{p+q} \left( \sum_{i+j=k} a_i \alpha^i(b_j) \right) x^k.$$

Similarly, $R[[x, \alpha]]$ is a ring. The rings $R[x, \alpha]$ and $R[[x, \alpha]]$ are called the skew polynomial extension and the skew power series extension of $R$, respectively.

In [20], the ring $R$ is called Armendariz if for any $f(x) = \sum_{i=0}^{p} a_i x^i$, $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x, \alpha]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all $i$ and $j$. This definition of Armendariz ring is extended to module in [18]. A module $M$ is called $\alpha$-Armendariz if the following conditions (i) and (ii) are satisfied, and the module $M$ is called $\alpha$-Armendariz ring of power series wise if the following conditions (i) and (iii) are satisfied.

(i) For $m \in M$ and $q \in R$, $ma = 0$ if and only if $m\alpha(a) = 0$.

(ii) For any $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x, \alpha]$, $f(x) = \sum_{j=0}^{q} a_j x^j \in R[x, \alpha]$,

$$m(x)f(x) = 0$$

implies $m_i \alpha_i(a_j) = 0$ for all $i$ and $j$.

(iii) For any $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[x, \alpha]$, $f(x) = \sum_{j=0}^{q} a_j x^j \in R[[x, \alpha]]$,

$$m(x)f(x) = 0$$

implies $m_i \alpha_i(a_j) = 0$ for all $i$ and $j$. 


In this paper, the ring $R$ is called $\alpha$-Armendariz (\(\alpha\)-Armendariz ring of power series wise) if $R_R$ is $\alpha$-Armendariz (\(\alpha\)-Armendariz ring of power series wise) module.

In [4], it was proved that a ring $R$ is quasi-Baer if and only if $R[X]$ is quasi-Baer iff $R[[X]]$ is quasi-Baer. Where $X$ is an arbitrary non-empty set of not necessarily commuting indeterminates. If $R$ is a reduced ring, then $R$ is Baer iff $R[x]$ is Baer iff $R[[x]]$ is Baer. In [7], the author showed that $R$ is right p.q. Baer ring iff $R[[x, \alpha]]$ is right p.q. Baer ring. In this paper, we investigated the equivalent relation between right p.q. Baer ring $R$ and right p.q. Baer ring $R[[x, \alpha]]$.

\section{Main Results}

**Lemma 2.1** ([3, Lemma 2.3]) Let $R$ be a ring such that for any $a, b \in R$, $ab = 0 \Rightarrow aRb = 0$, then $\alpha(e) = e$ for every idempotent $e \in R$.

**Proof** See [3, Lemma 2.3].

**Definition 2.2** Let $\alpha$ be an endomorphism of a ring $R$. A ring $R$ is called $\alpha$-Armendariz ring of power series wise if for $f(x) = \sum_{i=0}^{\infty} f_i x^i$, $g(x) = \sum_{j=0}^{\infty} g_j x^j \in R[[x, \alpha]]$, $f(x)g(x) = 0$ then $f_i \alpha^i(g_j) = 0$ or $f_i g_j = 0$ for all $i$ and $j$.

**Lemma 2.3** Let $R$ be an $\alpha$-Armendariz ring of power series wise. Then the set of all idempotents of $R[[x, \alpha]]$ coincides with the set of all idempotents of $R$ and $R[[x, \alpha]]$ is abelian ring.

**Proof** See [3, Lemma 3.5].

**Theorem 2.4** Let $R$ be an $\alpha$-Armendariz ring of power series wise. If $R[[x, \alpha]]$ is right p.q. Baer then $R$ is right p.q. Baer ring.

**Proof** Let any element $a \in R$. Since $R[[x, \alpha]]$ is right p.q. Baer, there exists an idempotent $e(x) \in R[[x, \alpha]]$ such that $r_R[[x, \alpha]](aR[[x, \alpha]]) = e(x)R[[x, \alpha]]$. So by Lemma 2.3, $r_R[[x, \alpha]](aR[[x, \alpha]]) = e_0 R[[x, \alpha]]$. Then $aR[[x, \alpha]]e_0 = 0$ implies $aRe_0 = 0$. Thus $e_0 R \subseteq r_R(aR)$. Again take any element $b \in r_R(aR)$. Since $R$ is $\alpha$-Armendariz ring of power series wise, so $b \in r_R[[x, \alpha]](aR[[x, \alpha]])$. By hypothesis $R[[x, \alpha]]$ is right p.q. Baer so $b = e(x)b$ and $b = e_0 b \in e_0 R$. Thus $r_R(aR) \subseteq e_0 R$. consequently $r_R(aR) = e_0 R$. Therefore $R$ is right p.q. Baer.
Example 2.5 ([13, p. 225]) For a field $F$, let

$$R = \left\{ (a_n) \in \prod_{n=1}^{\infty} F_n \mid a_n \text{ is eventually constant} \right\}.$$  

which is a subring of $\prod_{n=1}^{\infty} F_n$, where $f_n = f$ for $n = 12$. Then $R$ is commutative von-Neumann regular ring and hence it is a right p.q. Baer. Let $\alpha$ be the identity map on $R$. Then $R$ is $\alpha$-rigid ring (or $\alpha$-Armendariz ring of power series type). But $R[[x, \alpha]]$ is not p.q. Baer.

Definition 2.6 ([19, Z. Liu]) Let $\{e_0, e_1, e_2, \ldots\}$ be a countable family of idempotents of $R$. We say $\{e_0, e_1, e_2, \ldots\}$ has a generalized join in $I(R)$ if there exists an idempotent $e \in I(R)$ such that

(i) $e_i R(1 - e) = 0$

(ii) If $f \in I(R)$ is such that $e_i R(1 - f) = 0$. Then $e R(1 - f) = 0$.

Theorem 2.7 Let $R$ be an $\alpha$-Armendariz ring of power series wise. Then the following are equivalent.

(1) $R[[x, \alpha]]$ is right p.q. Baer

(2) $R$ is right p.q. Baer and any countable family of idempotents of $R$ has generalized join in $I(R)$.

Proof Let $R[[x, \alpha]]$ is right p.q. Baer then $R$ is right p.q. Baer by the Theorem 2.4.

Now only show that a countable family of idempotent $\{e_0, e_1, e_2, \ldots\}$ of $R$ has generalized join in $I(R)$.

Let $\tau(x) = e_0 + e_1 x + e_2 x^2 + \ldots \in R[[x, \alpha]]$. Since $R[[x, \alpha]]$ is p.q. Baer so there exists an idempotent $e(x) = \varepsilon_0 + \varepsilon_0 x + \varepsilon_0 x^2 + \ldots \in R[[x, \alpha]]$, where $\varepsilon_0, \varepsilon_1, \varepsilon_2 \ldots$ be an idempotents of $R$, such that $r_{R[[x, \alpha]]}(\tau(x) R[[x, \alpha]]) = e(x) R[[x, \alpha]]$. Since $R$ is $\alpha$-Armendariz ring of power series wise so by Lemma 2.3

$$r_{R[[x, \alpha]]}(\tau(x) R[[x, \alpha]]) = \varepsilon_0 R[[x, \alpha]].$$

i.e. $\tau(x) R[[x, \alpha]] \varepsilon_0 = 0$ implies $\tau(x) R \varepsilon_0 = 0$

$$\Rightarrow \tau(x) r \varepsilon_0 = 0 \quad \forall \ r \in R$$

$$\Rightarrow (e_0 + e_1 x + e_2 x^2 + \ldots) r \varepsilon_0 = 0$$

$$\Rightarrow e_0 r \varepsilon_0 + e_1 \alpha (r \varepsilon_0) x + e_2 \alpha^2 (r \varepsilon_0) + \ldots = 0$$

Since $R$ is $\alpha$-Armendariz ring of power series wise so $e_i r \varepsilon_0 = 0$ for every $i = 0, 1, 2, \ldots$. Set $\eta = 1 - \varepsilon_0$ so $\eta^2 = \eta \in R$. Thus $e_i r (1 - \eta) = 0$ for an
idempotent $\eta$. Suppose $g$ be any idempotent of $R$ such that $e_iR(1-g) = 0$ implies $e_ir (1-g) = 0$ for every $r \in R$. Then $e_ir = e_ir g$ for each $r \in R$. Let any $f(x) = \sum_{j=0}^{\infty} f_jx^j \in R[[x, \alpha]]$ and any $r \in R$, 

$$\tau(x)f(x)r(1-g) = (e_0 + e_1x + e_2x^2 + \ldots)(f_0 + f_1x + f_2x^2 + \ldots)r(1-g) = e_0f_0r(1-g) + (e_0f_1\alpha(r(1-g)) + e_1\alpha(f_0r(1-g)))x + \ldots$$

Since $e_iR(1-g) = 0$ and $R$ is $\alpha$-Armendariz ring of power series wise so,

$$\tau(x)f(x)r(1-g) = 0.$$ Thus $r(1-g) \in r_{R[[x, \alpha]]}(\tau(x)R[[x, \alpha]])$. Since $R[[x, \alpha]]$ is right p.q. Baer so,

$$r(1-g) \in r_{R[[x, \alpha]]}(\tau(x)R[[x, \alpha]]) = \varepsilon_0 R[[x, \alpha]].$$

Thus $r(1-g) = \varepsilon_0 r(1-g) \Rightarrow (1 - \varepsilon_0)r(1-g) = 0$. By setting $\eta = 1 - \varepsilon_0, \eta r(1-g) = 0$. Therefore $\eta$ is generalized join the set of idempotents $\{e_0, e_1, e_2, \ldots\}$. Let $f(x) = f_0 + f_1x + f_2x^2 + \ldots R[[x, \alpha]]$. Then $f_0, f_1, f_2, \ldots R$. Since $R$ is right p.q. Baer so there exists $e_i \in R \forall i = 0, 1, 2, \ldots$ such that $r_{R}(f,R) = e_iR$. By assumption the set $\{1 - e_i|i = 0, 1, 2, \ldots\}$ has generalized join $\eta$. Thus $(1 - e_i)R(1-\eta) = 0$. Now for any $g(x) = g_0 + g_1x + g_2x^2 + \ldots \in R[[x, \alpha]]$. Take

$$f(x)g(x)(1-\eta) = (f_0 + f_1x + f_2x^2 + \ldots)(g_0 + g_1x + g_2x^2 + \ldots)(1-\eta)$$

$$= \left(\sum_{i=0}^{\infty} f_ix^i\right)\left(\sum_{j=0}^{\infty} g_jx^j\right)(1-\eta)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{i+j=k} f_i\alpha^i(q_i)\alpha^j(1-\eta)\right)x^k$$

$$= \sum_{k=0}^{\infty} \left(\sum_{i+j=k} f_i\alpha^i(q_j)(1-\eta)\right)x^k$$

Since $\{1 - e_i|i = 0, 1, 2, \ldots\}$ has join $\eta$. So $e_i$ has join $(1-\eta)$. So $e_iR(1-\eta)$ and $R$ is $\alpha$-Armendariz ring of power series wise.

Thus $f(x)g(x)(1-\eta) = 0 \Rightarrow (1 - \eta) \in r_{R[[x, \alpha]]}(f(x)R[[x, \alpha]])$. So $(1 - \eta)R[[x, \alpha]] \subseteq r_{R[[x, \alpha]]}(f(x)R[[x, \alpha]])$.

Again suppose that $h(x) = h_0 + h_1x + h_2x^2 + \ldots \in r_{R[[x, \alpha]]}(f(x)R[[x, \alpha]])$ so

$$f(x)R[[x, \alpha]]h(x) = 0.$$
implies $f(x)Rh(x) = 0$

\[
\Rightarrow f(x)rh(x) = 0 \quad \forall \ r \in R
\]

\[
\Rightarrow r \sum_{i=0}^{\infty} f_i x^i \left( \sum_{j=0}^{\infty} h_j x^j \right) = 0
\]

\[
\Rightarrow \sum_{k=0}^{\infty} \left( \sum_{i+j=k} f_i \alpha^i(rh_j) \right) x^k = 0
\]

Since $R$ is $\alpha$-Armendariz ring of power series wise so $f_i \alpha^i(rh_j) = 0 \Rightarrow f_i rh_j = 0$ implies $h_j \in r_R(f_i R)$. Since $R$ is right p.q. Baer so $h_j \in r_R(f_i R) = e_i R \Rightarrow h_j = e_i h_j \forall i$ and $j$. Suppose that $r_R(h_j R) = \eta_j R$, where $\eta_j$ is an idempotents of $R$. Since $R$ is $\alpha$-Armendariz ring of power series wise so by [3, Corollary 2.21] $R$ is $\alpha$-abelian ring. Therefore $e_i$ is central idempotent. Thus $h_j r = e_i h_j r = h_j re_i$ implies $h_j r(1-e_i) = 0$. So $(1-e_i) \in r_R(h_j R) = \eta_j R$. Then $(1-e_i) = \eta_j(1-e_i)$ for every $i$ and $j$. So $(1-e_i) R(1-\eta_j) = 0$. But $\eta$ has generalized join $\{1-e_i|i = 0, 1, 2, \ldots\}$. Thus $\eta R(1-\eta_j) = 0$ for every $i$ and $j$. Hence $h_j = h_j - h_j \eta_j = h_j (1-\eta_j) = (1-\eta_j) h_j = (1-\eta)(1-\eta_j) h_j \in (1-\eta) R$

\[
\Rightarrow h_j = (1-\eta)(1-\eta_j) h_j \in (1-\eta) R
\]

\[
\Rightarrow h(x) \in (1-\eta) R[[x, \alpha]].
\]

Thus $r_R[[x, \alpha]](f(x)R[[x, \alpha]]) \subseteq (1-\eta) R[[x, \alpha]]$. Therefore $R[[x, \alpha]]$ is right p.q. Baer ring.

**Corollary 2.8 ([10, Theorem 3])** If $R$ is a ring then $R[[x]]$ is reduced p.p.-ring iff $R$ is reduced p.p.-ring any countable family of idempotents in $R$ has join in $I(R)$.

**References**


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