International Mathematical Forum, 5, 2010, no. 59, 2929 - 2943

Iterative Method for a Generalized Equilibrium Problem and Fixed Point Problem of Nonexpansive Mappings

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Abstract

In this paper, we introduce an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Then, we prove strong convergence theorems for nonexpansive mapping to solve a unique solution of the variational inequality. The results extended and improved the corresponding results of Y. Shehu [Fixed point solutions of generalized equilibrium problems for nonexpansive mappings, J. Com. Appl. Math. doi:10.1016/j.cam.2010.01.055], and many others.

Mathematics Subject Classification: 47H10, 47H09, 46B20

Keywords: β-inverse-strongly monotone; variational inequality; generalized equilibrium problems

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the real numbers. The equilibrium problem for F is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all} \quad y \in C. \tag{1}$$

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The set of solutions of (1) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1). In 1997, Combettes and Hirstoaga [5] introduced an iterative scheme of finding the best approximation to the innitial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

Let $A : C \to H$ be a nonlinear mapping. Then, we consider the following equilibrium problem: Find

$$z \in C \text{ such that } F(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C \tag{2}$$

The set of such $z \in C$ is denoted by EP, i.e.,

$EP = \{z \in C : F(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C\}$.

In the case of $A \equiv 0$, EP is denoted by $EP(F)$. In the case of $F \equiv 0$, EP is also denoted by $VI(C, A)$. A mapping T of C into itself is *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

A mapping $T : C \to C$ is said to be k-strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2,$$

for all $x, y \in C$. We see that, if $k = 0$ then mapping T is nonexpansive. A mapping $A : C \to H$ is an β-*inverse* $-$ *strongly monotone* mapping if

$$\langle Ax - Ay, x - y\rangle \geq \beta\|Ax - Ay\|^2, \text{ for all } x, y \in C$$

The set of fixed points of S is denoted by $F(T)$. Recently Tada and Takahashi [13], and Takahashi and Takahashi [12] considered iterative methods for finding an element of $EP(F) \cap F(T)$. On the other hand, Takahashi and Toyoda [14] introduced an iterative method for finding an element of $VI(C, A) \cap F(T)$, where A is an β-*inverse* $-$ *strongly monotone* mapping. Very recently, Moudafi [7] introduced an iterative method for finding an element of $EP \cap F(T)$, where $A : C \to H$ is an inverse-strongly monotone mapping and then proved a weak convergence theorem.

In 2010, Y. Shehu, [11], introduced an iterative scheme generated by $x_1 \in K$,

$$\begin{cases} F(x, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T u_n, n \geq 1 \end{cases} \tag{3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$,$\{\gamma_n\}_{n=1}^{\infty} \subset (0, 1)$ and $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$, then $\{x_n\}$ converges strongly to the common elements of the set of nonexpansive mapping and the set of a generalized equilibrium problems.

Motivates by (3.1), we introduce an iterative viscosity for finding common element of the set of fixed points of nonexpansive and the set of solution of a generalized equilibrium problems in Hilbert space. We show strong convergence of iterative viscosity to a common elements of the two sets.

2 Preliminary

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \text{ for all } y \in C. \tag{4}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{5}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{6}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{7}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda Au), \lambda > 0. \tag{8}$$

It is also known that H satisfies the *Opial's condition* [8], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\text{liminf}_{n\to\infty}\|x_n - x\| < \text{liminf}_{n\to\infty}\|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Hilbert space H, satisfies the Opial's property.

Remark We note that if A is a α-inverse-strongly monotone, for all $u, v \in C$ and $\lambda_n > 0$,

$$\begin{aligned}\|(I - \lambda_n A)u - (I - \lambda_n A)v\|^2 &= \|(u - v) - \lambda_n(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda_n\langle u - v, Au - Av\rangle \\ &\quad +\lambda_n^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au - Av\|^2. \end{aligned} \tag{9}$$

So, if $\lambda_n \leq 2\alpha$, then $I - \lambda_n A$ is a nonexpansive mapping from C to H.

Next, we collect some lemmas which will be used in the proof for the main result in this section.

Lemma 2.1 *Let H be a real Hilbert space. Then for any $x, y \in H$ we have*
(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$
(ii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$
(iii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \ \forall t \in [0,1]$.

Lemma 2.2 *[15] Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property,*

$$a_{n+1} \leq (1-\gamma_n)a_n + b_n, \ n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$, and $\{b_n\}$ is a sequence in $\mathbf{R}$ such that:
i) $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;
ii) $\limsup_{n\to\infty} \frac{b_n}{\gamma_n} \leq 0$ *or* $\Sigma_{n=1}^{\infty}|b_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 *[9] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbf{R}$, let us assume that F satisfies the following condition:
(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y \in C$,

$$\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.
The following lemma appears implicitly in [2].

Lemma 2.4 *[2] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \text{for all } y \in C.$$

The following lemma was also given in [3].

Lemma 2.5 *[5] Assume that $F : C \times C \to \mathbf{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping*

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $z \in H$. Then, the following hold:

1. T_r is single-valued;

2. T_r is firmly nonexpansive, i. e., $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$ for all $x, y \in H$;

3. $F(T_r) = EP(F)$;

4. $EP(F)$ is closed and convex.

We can also obtain the following lemma

Lemma 2.6 *[12] Let C, H and $T_{r_n}x$ be as in Lemma 2.5. Then the following holds:*

$\|T_s x - T_t x\|^2 \leq \frac{s-t}{s}\langle T_s x - T_t x, T_s x - x\rangle$

for all $s, t > 0$ and $x \in H$.

Lemma 2.7 *[10] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

3 Main Results

Theorem 3.1 *Let C be a closed convex nonempty subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ satisfying (A1)-(A4), A be an α-inverse-strongly monotone mapping of C into H, T be a nonexpansive mapping of C into itself and let f be contraction mapping for some $\alpha \in (0,1)$. Suppose $F(T) \cap EP \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be generated by $x_1 \in K$,*

$$\begin{cases} F(x,y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T u_n, n \geq 1 \end{cases} \tag{10}$$

where $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$,$\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{r_n\}_{n=1}^{\infty} \subset [0, 2\beta]$. If $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are chosen so that $\{r_n\}_{n=1}^{\infty} \subset [a,b]$ for some a,b with $0 < a < b < 2\beta$, $\alpha_n + \beta_n + \gamma_n = 1, n \geq 1$

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \tag{11}$$

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \lim_{n\to\infty} |r_{n+1} - r_n| = 0 \tag{12}$$

then,$\{x_n\}_{n=1}^{\infty}$ converges strongly to $z_o = P_{F(T)\cap EP} f(z_0)$.

Proof. Let $F(T)\cap EP\neq\emptyset$. Put $u_n := T_{r_n}(x_n - r_nAx_n), n\geq 1$. let $x^*\in F(T)\cap EP$ and $\{T_{r_n}\}_{n=1}^{\infty}$ be a sequence of mapping defined as in Lemma 2.5. Since T_{r_n} is nonexpansive and $x^* = T_{r_n}(x^* - r_nAx^*)$. From (9), we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - x^*\|^2\\
&= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(x^* - r_nAx^*)\|^2\\
&\leq \|(I - r_nA)x_n - (I - r_nA)x^*\|^2\\
&\leq \|x_n - x^*\|^2 + r_n(r_n - 2\beta)\|Ax_n - Ax^*\|^2 \qquad (13)\\
&\leq \|x_n - x^*\|^2 (\text{ since } r_n < 2\beta, \forall n\geq 1). \qquad (14)
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_nf(x_n) + \beta_nx_n + \gamma_nTu_n - x^*\|\\
&= \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(Tu_n - x^*)\|\\
&\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|Tu_n - x^*\|\\
&\leq \alpha_n(\|f(x_n) - f(x^*)\| + \|f(x^*) - x^*\|) + \beta_n\|x_n - x^*\|\\
&\quad +\gamma_n\|u_n - x^*\|\\
&\leq \alpha_n\alpha\|x_n - x^*\| + \alpha_n\|f(x)^* - x^*\| + \beta_n\|x_n - x^*\|\\
&\quad +\gamma_n\|x_n - x^*\|\\
&= (1 - \alpha_n(1 - \alpha))\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\|\\
&\leq \max\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{(1 - \alpha)}\}\\
&\ \vdots\\
&\leq \max\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha}\}
\end{aligned} \qquad (15)$$

Hence,$\{x_n\}_{n=1}^{\infty}$ is bounded and so, $\{u_n\}_{n=1}^{\infty}$, $\{Tu_n\}_{n=1}^{\infty}$, $\{Ax_n\}_{n=1}^{\infty}$ and $\{f(x_n)\}_{n=1}^{\infty}$ are also bounded. Put $w_n = (x_n - r_nAx_n)$ and from Lemma 2.6, we have

$$\begin{aligned}
\|T_{r_{n+1}}w_{n+1} - T_{r_n}w_{n+1}\|^2 &\leq \frac{r_{n+1} - r_n}{r_{n+1}}\langle T_{r_{n+1}}w_{n+1} - T_{r_n}w_{n+1}, T_{r_{n+1}}w_{n+1} - w_{n+1}\rangle\\
&\leq \frac{|r_{n+1} - r_n|}{a}\|T_{r_{n+1}}w_{n+1} - T_{r_n}w_{n+1}\|\|T_{r_{n+1}}w_{n+1} - w_{n+1}\|
\end{aligned}$$

and hence

$$\begin{aligned}
\|T_{r_{n+1}}w_{n+1} - T_{r_n}w_{n+1}\| &\leq \frac{|r_{n+1} - r_n|}{a}\|T_{r_{n+1}}w_{n+1} - w_{n+1}\|\\
&\leq \frac{|r_{n+1} - r_n|}{a}M, \qquad (16)
\end{aligned}$$

where $M = \sup\{\|T_{r_n}w_n - w_n\| : n \in \mathbf{N}\} < \infty$. From $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, $\lim_{n\to\infty} \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_{n+1}\| = 0$. Consider,

$$\begin{aligned}
\|w_{n+1} - w_n\|^2 &= \|(x_{n+1} - r_{n+1}Ax_{n+1}) - (x_n - r_nAx_n)\|^2 \\
&\leq (\|(x_{n+1} - r_{n+1}Ax_{n+1}) - (x_{n+1} - r_{n+1}Ax_n) \\
&\quad + \|(x_{n+1} - r_{n+1}Ax_n) - (x_{n+1} - r_nAx_n)\|)^2 \\
&\leq (\|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\|)^2 \\
&= \|x_{n+1} - x_n\|^2 + 2\|x_{n+1} - x_n\|(|r_{n+1} - r_n|\|Ax_n\|) \\
&\quad + (|r_{n+1} - r_n|\|Ax_n\|)^2 \\
&\leq \|x_{n+1} - x_n\|^2 + c_n,
\end{aligned}$$

where $c_n = |r_{n+1} - r_n|(2\|x_{n+1} - x_n\|\|Ax_n\| + (|r_{n+1} - r_n|\|Ax_n\|)^2)$. Since $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, we have $\lim_{n\to\infty} c_n = 0$, and hence

$$\begin{aligned}
\|u_{n+1} - u_n\|^2 &= \|T_{r_{n+1}}w_{n+1} - T_{r_n}w_n\|^2 \\
&\leq (\|T_{r_{n+1}}w_{n+1} - T_{r_n}w_{n+1}\| + \|T_{r_n}w_{n+1} - T_{r_n}w_n\|)^2 \\
&\leq \|T_{r_{n+1}}w_{n+1} - T_{r_n}w_{n+1}\|^2 + \|w_{n+1} - w_n\|^2 \\
&\quad + 2\|T_{r_{n+1}}w_{n+1} - T_{r_n}w_{n+1}\|\|T_{r_n}w_{n+1} - T_{r_n}w_n\| \\
&\leq \|x_{n+1} - x_n\|^2 + d_n,
\end{aligned}$$

where $d_n = c_n + \|T_{r_{n+1}}w_{n+1} - T_{r_n}w_{n+1}\|\{\|T_{r_{n+1}}w_{n+1} - T_{r_n}w_{n+1}\| + 2\|T_{r_n}w_{n+1} - T_{r_n}w_n\|\}$, then $\lim_{n\to\infty} d_n = 0$. There exists $N \in \mathbf{N}$ such that

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\|, \quad \text{for all } n \geq N. \tag{17}$$

Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$. Then

$$z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n f_(x_n) + \gamma_n Tu_n}{1 - \beta_n}$$

Hence, we obtain

$$\begin{aligned}
z_{n+1} - z_n &= \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}Tu_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f_(x_n) + \gamma_n Tu_n}{1 - \beta_n} \\
&= (\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n})f(x_{n+1}) + \frac{\alpha_n}{1 - \beta_n}(f(x_{n+1}) - f(x_n)) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(Tu_{n+1} - Tu_n) + (\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n})Tu_n
\end{aligned} \tag{18}$$

From (18) and (17), we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq |\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}|\|f(x_{n+1})\| + \frac{\alpha_n}{1 - \beta_n}\|f(x_{n+1}) - f(x_n)\| \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\|Tu_{n+1} - Tu_n\| + |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}|\|Tu_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq \left|\frac{\alpha_n}{1-\beta_n} - \frac{\alpha_n}{1-\beta_n}\right|\|f(x_{n+1})\| + \frac{\alpha_n \alpha}{1-\beta_n}\|x_{n+1} - x_n\| \\
&\quad + \left(\frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}}\right)\|u_{n+1} - u_n\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|Tu_n\| \\
&\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|f(x_{n+1})\| + \frac{\alpha_n \alpha}{1-\beta_n}\|x_{n+1} - x_n\| \\
&\quad + \|x_{n+1} - x_n\| - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|Tu_n\|
\end{aligned}$$

It follow that

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|f(x_{n+1})\| \\
&\quad + \frac{\alpha_n \alpha}{1-\beta_n}\|x_{n+1} - x_n\| \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|Tu_n\|
\end{aligned}$$

This together with conditions $\lim_{n\to\infty} \alpha_n = 0$ and boundedness, we have

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

From $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and by Lemma 2.7, we get

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{19}$$

Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|z_n - x_n\| = 0.$$

Observe that

$$\begin{aligned}
x_{n+1} - x_n &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n T u_n - x_n \\
&= \alpha_n (f(x_n) - x_n) + \gamma_n (Tu_n - x_n)
\end{aligned}$$

Then, it follows from$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ which implies that that

$$\lim_{n\to\infty} \|Tu_n - x_n\| = 0. \tag{20}$$

From (10) and (13), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n T u_n - x^*\|^2 \\
&= \|\alpha_n (f(x_n) - x^*) + \beta_n (x_n - x^*) + \gamma_n (Tu_n - x^*)\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|Tu_n - x^*\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|u_n - x^*\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n [\|x_n - x^*\|^2 \\
&\quad + r_n (r_n - 2\beta)\|Ax_n - Ax^*\|^2] \\
&= \alpha_n \|f(x_n) - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\
&\quad + \gamma_n r_n (r_n - 2\beta)\|Ax_n - Ax^*\|
\end{aligned}$$

Therefore, we have

$$\begin{aligned} -\gamma_n a(b-2\beta_n)\|Ax_n - Ax^*\|^2 &\le \alpha_n\|f(x_n)-x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1}-x^*\|^2 \\ &\le \alpha_n\|f(x_n)-x^*\|^2 \\ &\quad + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1}-x^*\|) \end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0, \lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and boundednecs, we obtain

$$\lim_{n\to\infty}\|Ax_n - Ax^*\| = 0$$

By Lemma 2.5 we obtain

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - x^*\|^2 \\ &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(x^* - r_nAx^*)\|^2 \\ &\le \langle (I-r_nA)x_n - (I-r_nA)x^*, u_n - x^*\rangle \\ &= \frac{1}{2}[\|(I-r_nA)x_n - (I-r_nA)x^*\|^2 + \|u_n - x^*\|^2 - \|(I-r_nA)x_n \\ &\quad -(I-r_nA)x^* - (u_n - x^*)\|^2] \\ &\le \frac{1}{2}[\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 - r_n(Ax_n - Ax^*)\|^2] \\ &= \frac{1}{2}[\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n\langle x_n - u_n, Ax_n - Ax^*\rangle - r_n^2\|Ax_n - Ax^*\|^2] \end{aligned}$$

Thus,

$$\begin{aligned} \|u_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Ax^*\rangle \\ &\quad - r_n^2\|Ax_n - Ax^*\|^2 \end{aligned}$$

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n Tu_n - x^*\|^2 \\ &\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|Tu_n - x^*\|^2 \\ &\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\ &\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n[\|x_n - x^*\|^2 \\ &\quad - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Ax^*\rangle - r_n^2\|Ax_n - Ax^*\|^2] \\ &\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 \\ &\quad - \gamma_n\|x_n - u_n\|^2 + 2\gamma_n\|x_n - u_n\|\|Ax_n - Ax^*\| \end{aligned}$$

and hence

$$\begin{aligned} \gamma_n\|x_n - u_n\|^2 &\le \alpha_n\|f(x_n)-x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1}-x^*\|^2 \\ &\quad + 2\gamma_n\|x_n - u_n\|\|Ax_n - Ax^*\| \\ &\le \alpha_n\|f(x_n)-x^*\|^2 + \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1}-x^*\|) \\ &\quad + 2\gamma_n\|x_n - u_n\|\|Ax_n - Ax^*\| \end{aligned}$$

By $\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \lim_{n\to\infty} \|Ax - Ax^*\| = 0$ and boundednecs,

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{21}$$

From $\lim_{n\to\infty} \|Tu_n - x_n\| = 0$ and $\lim_{n\to\infty} \|u_n - x_n\| = 0$, this implies that

$$\|Tu_n - u_n\| \leq \|Tu_n - x_n\| + \|x_n - u_n\| \to 0 asn \to \infty$$

Next,we show that

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle \leq 0, \tag{22}$$

where $z_0 = P_{F(T)\cap EP} f(z_0)$. There exists subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, Tu_n - z_0\rangle = \lim_{i\to\infty} \langle f(z_0) - z_0, Tu_{n_i} - z_0\rangle. \tag{23}$$

Since $\{u_n\}$ is bounded there exists a subsequence $\{u_{n_{i_k}}\}$ of $\{u_{n_i}\}$ converges weakly to z. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup z$. Since $\lim_{n\to\infty} \|Tu_n - u_n\| = 0$, we obtain $Tu_{n_i} \rightharpoonup z$. We must show that $z \in F(T) \cap EP$. Let us show $z \in EP$. It follows by (10) and (A2) that

$$\langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n),$$

and hence

$$\langle Ax_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}). \tag{24}$$

Put $y_t = ty + (1-t)z$ for all $t \in (0, 1]$ and $y \in C$. Then we have $y_t \in C$. So, from (24), we have

$\langle y_t - u_{n_i}, Ay_t\rangle - \langle y_t - u_{n_i}, Ay_t\rangle = 0 \geq -\langle y_t - u_{n_i}, Ax_{n_i}\rangle - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y, u_{n_i})$

and hence

$\langle y_t - u_{n_i}, Ay_t\rangle \geq \langle y_t - u_{n_i}, Ay_t\rangle - \langle y_t - u_{n_i}, Ax_{n_i}\rangle - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y, u_{n_i})$
$= \langle y_t - u_{n_i}, Ay_t - Au_{n_i}\rangle + \langle y_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y, u_{n_i})$.

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Au_{n_i} - Ax_{n_i}\| \to 0$. Further, from monotonicity of A, we have

$$\langle y_t - u_{n_i}, Ay_t - Au_{n_i}\rangle \geq 0.$$

So, from (A4) we have

$$\langle y_t - z, Ay_t\rangle \geq F(y_t, z), \tag{25}$$

as $i \to \infty$. From (A1), (A4) and (25), we have

$$\begin{aligned}0 = F(y_t, y_t) &\le tF(y_t, y) + (1-t)F(y_t, z)\\ &\le tF(y_t, y) + (1-t)\langle y_t - z, Ay_t\rangle\\ &\le tF(y_t, y) + (1-t)t\langle y - z, Ay_t\rangle\end{aligned}$$

and hence

$$0 \le F(y_t, y) + (1-t)\langle y - z, Ay_t\rangle.$$

Letting $t \to 0$, we have for each $y \in C$,

$$0 \le F(z, y) + \langle y - z, Az\rangle.$$

This implies that $z \in EP$. Assume $z \neq Tz$, by Opials condition, we have

$$\begin{aligned}\liminf_{i\to\infty} \|u_{n_i} - z\| &< \liminf_{i\to\infty} \|u_{n_i} - T_z\|\\ &\le \liminf_{i\to\infty} \|u_{n_i} - Tu_{n_i} + Tu_{n_i} - T_z\|\\ &\le \liminf_{i\to\infty} \|u_{n_i} - z\|\end{aligned}$$

a contradiction, hence $z \in F(T)$. From (2.2), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0\rangle &= \limsup_{n\to\infty}\langle f(z_0) - z_0, Tu_n - z_0\rangle\\ &= \limsup_{i\to\infty}\langle f(z_0) - z_0, Tu_{n_i} - z_0\rangle\\ &= \langle f(z_0) - z_0, z - z_0\rangle \le 0 \qquad (26)\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n Tu_n - z_0, x_{n+1} - z_0\rangle\\ &= \alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle + \beta_n\langle x_n - z_0, x_{n+1} - z_o\rangle\\ &\quad +\gamma_n\langle Tu_n - z_0, x_{n+1} - z_0\rangle\\ &\le \frac{1}{2}\beta_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2)\\ &\quad +\alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle\\ &\quad +\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle + \gamma_n\|Tu_n - z_0\|\|x_{n+1} - z_0\|\\ &\le \frac{1}{2}\beta_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n\alpha\|x_n - z_0\|\|x_{n+1} - z_0\|\\ &\quad +\gamma_n\|x_n - z_0\|\|x_{n+1} - z_0\| + \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle\\ &\le \frac{1}{2}\beta_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2)\\ &\quad +\alpha_n\alpha\frac{1}{2}(\|x_n - z_0\| + \|x_{n+1} - z_0\|)^2\\ &\quad +\frac{1}{2}\gamma_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|) + \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle\\ &= \frac{1}{2}(\beta_n + \gamma_n + \alpha_n\alpha)(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2)\\ &\quad +\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle\end{aligned}$$

$$= \frac{1}{2}(1-\alpha_n(1-\alpha))(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\ +\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle$$

which in turn implies that

$$\begin{aligned} \|x_{n+1} - z_0\|^2 &\leq (1-\alpha_n(1-\alpha))\|x_n - z_0\|^2 + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &= (1-\alpha_n(1-\alpha))\|x_n - z_0\|^2 \\ &\quad +(\frac{2}{1-\alpha})\alpha_n(1-\alpha)(\langle f(z_0) - z_0, x_{n+1} - z_0\rangle) \end{aligned}$$

From (26) and Lemma 2.2, we have $\lim_{n\to\infty}\|x_n - z_0\| = 0$, we get that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)\cap EP}f(z_0)$.

4 Application

Theorem 4.1 *Let C be a closed convex nonempty subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ satisfying* (A1)-(A4)*, A be an α-inverse-strongly monotone mapping of C into H, T be a nonexpansive mapping of C into itself and suppose $F(T) \cap E(P) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be generated by $x_1 \in K$,*

$$\begin{cases} F(x,y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T u_n, n \geq 1 \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$,$\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$. If $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are chosen so that $\{r_n\}_{n=1}^{\infty} \subset [a,b]$ for some a,b with $0 < a < b < 2\alpha$, $\alpha_n + \beta_n + \gamma_n = 1, n \geq 1$

$$\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty \tag{27}$$

$$0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1, \lim_{n\to\infty}|r_{n+1} - r_n| = 0 \tag{28}$$

then,$\{x_n\}_{n=1}^{\infty}$ converges strongly to $z_o = P_{F(T)\cap EP}u$.

Proof. Taking $f(x) = u$ for all $x \in K$ in Theorem 3.1, we have the desired conclusion. ◇

Theorem 4.2 *Let K be a closed convex nonempty subset of a real Hilbert space H. Let F be a bifunction from $K \times K$ satisfying* (A1)-(A4)*, and let T be a nonexpansive mapping of K into itself let f be contraction mapping for some $\alpha \in (0,1)$. Suppose $F(T) \cap EP(F) \neq \emptyset$ and $u \in K$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be generated by $x_1 \in K$*

$$\begin{cases} F(u_n,y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in K \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T u_n, n \geq 1 \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$,$\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$ *and* $\{r_n\}_{n=1}^{\infty} \subset [0,2\alpha]$. *If* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *are chosen so that* $\{r_n\}_{n=1}^{\infty} \subset [a,b]$ *for some a,b with* $0 < a < b < 2\alpha$, $\alpha_n + \beta_n + \gamma_n = 1, n \geq 1$

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$$

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \lim_{n\to\infty} |r_{n+1} - r_n| = 0$$

then,$\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $z_o = P_{F(T)\cap EP(F)} f(z_0)$.

Proof. Taking $A \equiv 0$ in Theorem 3.1, we have the desired conclusion.

Theorem 4.3 *Let K be a closed convex nonempty subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of K into H and let T be a nonexpansive mapping of K into itself. Suppose* $F(T)\cap VI(K,A) \neq \emptyset$ *and* $u \in K$. *Let* $\{x_n\}_{n=1}^{\infty}$ *are generated by* $x_1 \in K$,

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T P_K(x_n - r_n A x_n), n \geq 1;$$

where $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$,$\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$ *and* $\{r_n\}_{n=1}^{\infty} \subset [0,2\alpha]$. *If* $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$,$\{\gamma_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *are chosen so that* $\{r_n\}_{n=1}^{\infty} \subset [a,b]$ *for some a,b with* $0 < a < b < 2\alpha$, $\alpha_n + \beta_n + \gamma_n = 1, n \geq 1$

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$$

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \lim_{n\to\infty} |r_{n+1} - r_n| = 0$$

then,$\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $z_o = P_{F(T)\cap VI(K,A)^u}$.

Proof. Taking $F(x,y) = 0, \forall x,y \in K$ and $f(x) = u$ for all $x \in K$ in Theorem 3.1, we have

$$\langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in K, \forall n \geq 1.$$

Thus

$$\langle y - u_n, x_n - r_n A x_n - u_n\rangle \leq 0, \forall y \in K, \forall n \geq 1.$$

This implies

$$P_K(x_n - r_n A x_n) = u_n, \forall \geq 1.$$

Hence, the desired conclusion follow from Theorem 3.1

Theorem 4.4 *Let K be a closed convex nonempty subset of a real Hilbert space H. Let F be a bifunction from $K \times K$ satisfying* (A1)-(A4)*, S be a k-strictly pseudocontractive mapping of K into itself and let T be a nonexpansive mapping of K into itself. Suppose $F(T) \cap EP \neq \emptyset$ and $u \in K$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be generated by $x_1 \in K$,*

$$\begin{cases} F(u_n, y) + \langle (I-S)x_n, y-u_n \rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n \rangle \geq 0, \forall y \in K \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T u_n, n \geq 1; \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$,$\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{r_n\}_{n=1}^{\infty} \subset [0, 1-k]$. If $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$,$\{\gamma_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are chosen so that $\{r_n\}_{n=1}^{\infty} \subset [a, b]$ for some a,b with $0 < a < b < 1-k$, $\alpha_n + \beta_n + \gamma_n = 1, n \geq 1$

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$$

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \lim_{n\to\infty} |r_{n+1} - r_n| = 0$$

then,$\{x_n\}_{n=1}^{\infty}$ converges strongly to $z_o = P_{F(S)\cap EP^u}$.

Proof. Put $A := I - S$. Then A is $\frac{(1-k)}{2}$-inverse-strongly monotone. We have $F(S) = VI(K, I)$ and $P_K(x_n - \lambda_n A x_n) = (1-\lambda_n)x_n + \lambda_n T x_n$ and $f(x) = u$ for all $x \in K$. Then, by Theorem 3.1, obtain the desired result.Remark 4.4, Lemma 2.3 of Takahashi and Takahashi [12] was not used in the proof process of our Theorem 3.1.

Acknowledgements. The authors wish to express their gratitude to the referees for a careful reading of the manuscript and helpful suggestions. This research is supported by the Centre of Excellence in Mathematics, the commission on Higher Education, Thailand.

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Received: May, 2010