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A Note on the Growth Properties of Composite Entire Functions Related to Proximate Order

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Abstract

In this paper we deduce a consequence of a result of Datta and Tamang[1] on the basis of proximate order of entire functions.

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1 Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The definitions of order ρ_f and proximate order $\rho_f(r)$ of entire f with finite order are well known. We do not explain the standard notations and definitions in the theory of entire functions as those are available in [2]. In the paper we deduce an application of a result of Datta and Tamang [1] using proximate order of entire functions of finite order.

2 Lemma.

In the section we present a lemma which will be needed in the sequel.

The following lemma is in fact a result of Datta and Tamang [1].

Lemma 1 *[1] Let f and g be two entire functions of finite lower order such that $0 < \lambda_g < \lambda_f$ and $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ for an $\varepsilon > 0$. Then for all sufficiently large values of r,*

$$T(r, f \circ g) < T(M(r,f), g) \text{ for all } r > 0.$$

3 Theorem.

In this section we present the main result of the paper.

Theorem 1 *Let f and g be two entire functions such that $0 < \lambda_g < \lambda_f \leq \rho_f < \infty$ and $\rho_g < \infty$. Also let $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ for an $\varepsilon > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,f)} \leq 3.\rho_g.2^{\rho_f}.$$

Proof. In view of Lemma 1 we obtain for all large values of r,

$$\begin{aligned} \log T(r, f\circ g) &\leq \log T(M(r,f), g) \\ i.e., \quad \log T(r, f\circ g) &\leq (\rho_g + \varepsilon)\log M(r,f) \\ i.e., \quad \liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,f)} &\leq (\rho_g+\varepsilon)\liminf_{r\to\infty}\frac{\log M(r,f)}{T(r,f)}. \end{aligned}$$

Since ε (> 0) is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,f)} \leq \rho_g \liminf_{r\to\infty}\frac{\log M(r,f)}{T(r,f)}. \tag{1}$$

As $\limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f(r)}} = 1$, for given ε $(0 < \varepsilon < 1)$ we get for all large values of r,

$$T(r,f) < (1+\varepsilon) r^{\rho_f(r)}$$

and for a sequence of values of r tending to infinity

$$T(r,f) > (1-\varepsilon) r^{\rho_f(r)}.$$

Since $\log M(r,f) \leq 3T(2r,f)$ for a sequence of values of r tending to infinity we get for any δ (> 0),

$$\frac{\log M(r,f)}{T(r,f)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)} . \frac{(2r)^{\rho_f+\delta}}{(2r)^{\rho_f+\delta-\rho_f(2r)}} . \frac{1}{r^{\rho_f(r)}}.$$

As $\frac{d}{dr}r^{\rho_f+\delta-\rho_f(r)} = \left\{\rho_f + \delta - \rho_f(r) - r\rho_f'(r)\log r\right\} r^{\rho_f+\delta-\rho_f(r)-1} > 0$ for all sufficiently large values of r, $(2r)^{\rho_f+\delta-\rho_f(2r)}$ is ultimately an increasing function of r and so it follows from above that

$$\frac{\log M(r,f)}{T(r,f)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)}.2^{\rho_f+\delta}. \tag{2}$$

Since ε (> 0) and δ (> 0) are arbitrary, we obtain from (2) that

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)} \leq 3.2^{\rho_f}. \tag{3}$$

Thus the theorem follows from (1) and (3). ■

Remark 1 *If in particular $g(0) = 0$ and the other conditions remain the same then also Theorem 1 holds.*

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