New Infinite Families of Orthogonal Designs Constructed from Complementary Sequences

Christos Koukouvinos
Department of Mathematics
National Technical University of Athens
Zografou 15773, Athens, Greece
ckoukouv@math.ntua.gr

Dimitris E. Simos
Department of Mathematics
National Technical University of Athens
Zografou 15773, Athens, Greece
dsimos@math.ntua.gr

Abstract

In this paper, we present new infinite families of three and four variable orthogonal designs based on several constructions derived from complementary sequences. The above method leads to the construction of many classes of orthogonal designs. In addition, we obtain new infinite families of weighing matrices constructed by complementary sequences, such as $W(144 + 4s, 144)$ and $W(224 + 4s, 196)$ for all $s \geq 0$. These families resolve the existence and construction of over 20 weighing matrices which are listed as open in the second edition of the Handbook of Combinatorial Designs.

Mathematics Subject Classification: 05B20, 62K05

Keywords: Complementary sequences, Directed sequences, Golay sequences, Non-periodic autocorrelation function, Orthogonal Designs, Weighing matrices

1 Preliminaries

An orthogonal design of order $n$ and type $(s_1, s_2, \ldots, s_k)$ denoted $OD(n; s_1, s_2, \ldots, s_k)$ in the commuting variables $x_1, x_2, \ldots, x_k$, is a square matrix $D$ of
order $n$ with entries from the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_k\}$ satisfying

$$DD^T = \sum_{i=1}^{k} (s_i x_i^2) I_n,$$

where $I_n$ is the identity matrix of order $n$. A weighing matrix $W = W(n, w)$ is a square matrix with entries $0, \pm 1$ having $w$ non-zero entries per row and column and inner product of distinct rows equal to zero. Therefore $W$ satisfies $WW^T = wI_n$. The number $w$ is called the weight of $W$. Orthogonal designs are used in Combinatorics, Statistics, Coding Theory, Telecommunications and other areas. For more details on orthogonal designs see [2]. An $OD(m; a_1, \ldots, a_k)$ will be called full orthogonal design, if $a_1 + \cdots + a_k = m$. We adopt and modify accordingly the following definitions from [2, 5] for the results of this paper:

**Definition 1.1.** Let $A = [a_1, a_2, \ldots, a_n]$ be a sequence of length $n$. The non-periodic autocorrelation function, NPAF, $N_A(s)$ is defined as:

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, s = 0, 1, \ldots, n-1.$$

**Definition 1.2.** The weight of the sequence $A$ is the number of nonzero entries in $A$. $m$ sequences $A_1, A_2, \ldots, A_m$ of length $n$ with entries $\{0, \pm 1\}$ are $m$-complementary sequences (CS) of weight $w$, written $m$-CS($n, w$) if $N_{A_1}(s) + N_{A_2}(s) + \ldots + N_{A_m}(s) = 0$, for $s = 1, \ldots, n-1$. Such sequences have zero NPAF.

**Definition 1.3.** $m$ sequences of length $n$ of commuting variables from the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_k\}$ with zero NPAF where $\pm x_i$ occurs $s_i$ times are called $m$-NPAF($n; s_1, s_2, \ldots, s_k$) sequences of type $(s_1, s_2, \ldots, s_k)$.

**Remark 1.4.** $2$-CS($n, 2n$) are Golay sequences (GS) of length $n$, denoted by GS($n$).

The following facts for the following sets of combinatorial objects are well known, and there exist maps (constructions) for CS:

$$2 - CS(n, w) \rightarrow W(2n, w)$$

$$2 - NPAF(n; u, v) \rightarrow OD(2n; u, v)$$

$$4 - CS(n, w) \rightarrow W(4n, w)$$

$$4 - NPAF(n; s_1, s_2, \ldots, s_k) \rightarrow OD(4n; s_1, s_2, \ldots, s_k)$$

$$m - NPAF(n; s_1, s_2, \ldots, s_k) \rightarrow m - CS(n, \sum_{i=1}^{k} s_i)$$
New infinite families of orthogonal designs

\[ m - CS(n, w) \to m - CS(n + s, w), \forall s \in \mathbb{N} \quad (6) \]
\[ m - NPAF(n; s_1, s_2, \ldots, s_k) \to m - NPAF(n + s; s_1, s_2, \ldots, s_k), \forall s \in \mathbb{N} \quad (7) \]
\[ GS(g) \to 2 - NPAF(g; 2g) \quad (8) \]
\[ GS(g) \to 2 - NPAF(2g; 2g, 2g) \quad (9) \]
\[ GS(g_1) \times GS(g_2) \to GS(g_1g_2) \quad (10) \]

For (1) and (2) use the two sequences in the two circulant construction of [6] and [2]. For constructions (3) and (4) use the four sequences as first rows of circulant matrices in the Goethals-Seidel array (see Geramita and Seberry [2, page 107]). For maps (6) and (7), the sequences are being padded at the end with \( s \) zeros to obtain longer sequences with zero NPAF, while for map (5) replace the commuting variables with \( \pm 1 \). We multiply the GS with a commuting variable \( \alpha \) to obtain map (8). In construction (9) suppose that the two GS\((g)\) are \( G \) and \( H \). Denote by \( G^* \) and \( H^* \) the reverse sequences of \( G \) and \( H \) respectively (where \( A^* = [a_n, \ldots, a_2, a_1] \) for the reversed sequence of \( A = [a_1, a_2, \ldots, a_n] \)). The symbol \( | \) denotes concatenation of sequences. Let \( x \) and \( y \) be commuting variables. Then the sequences \( P' = [Gx \mid Hy] \) and \( Q' = [H^*x \mid -G^*y] \) are \( 2 - NPAF(2g; 2g, 2g) \). Golay sequences are known for lengths \( n = 2 \) and \( 10 \) ([3]), and for length \( 26 \) ([4]). The composition construction (10) is due to Turyn [9]. Therefore \( GS(n) \), exist for \( n = 2^a.10^b.26^c \) where \( a, b, c \) nonnegative integers.

2 Orthogonal designs from complementary sequences

In this Section, we give some constructions of new infinite families of orthogonal designs from complementary sequences. For more details, regarding sequences with zero autocorrelation we refer to [5]. We note that for every construction given, assuming complementary sequences with no zeros exist, the case \( s = 0 \) generates full orthogonal designs.

2.1 Orthogonal designs from NPAF sequences

Theorem 2.1. Suppose there exist \( 2 - NPAF(n; u, v) \) and GS\((g)\). Then there exist the following maps (constructions) for CS and orthogonal designs:

(i) \( 2 - NPAF(n; u, v) \times GS(g) \to 4 - NPAF(n + 2g + s; 2u, 2v, 4g, 4g) \forall s \in \mathbb{N} \)
(ii) \( 2 - NPAF(n; u, v) \times GS(g) \to OD(4 \cdot (n + 2g + s); 2u, 2v, 4g, 4g) \forall s \in \mathbb{N} \)
Proof. (i) Let \( A = [a_1, \ldots, a_n] \) and \( B = [b_1, \ldots, b_n] \) where \( a_k, b_k \in \{0, \pm \alpha, \pm \beta\}, \quad k = 1, \ldots, n \), be \( 2 - \text{NPAF}(n; u, v) \rightarrow \text{OD}(2n; u, v) \) (apply map (2)).

Now suppose that the two \( \text{GS}(g) \) are \( G \) and \( H \). Using map (9), \( P' = [Gx \mid Hy] \) and \( Q' = [H^*x \mid -G^*y] \) are \( 2 - \text{NPAF}(2g; 2g, 2g) \).

These two pairs are doubled accordingly and the following four sequences are formed,

\[
\begin{align*}
P &= [a_1, \ldots, a_n \mid Gx \mid Hy] \\
Q &= [a_1, \ldots, a_n \mid -Gx \mid -Hy] \\
R &= [b_1, \ldots, b_n \mid H^*x \mid -G^*y] \\
S &= [b_1, \ldots, b_n \mid -H^*x \mid G^*y]
\end{align*}
\]

(11)

The sequences \( P, Q, R, S \) have zero NPAF by a lengthy but straightforward calculation of their autocorrelation function and therefore are \( 4 - \text{NPAF}(n + 2g; 2u, 2v, 4g, 4g) \). Applying map (7) to \( P, Q, R, S \) we obtain \( 4 - \text{NPAF}(n + 2g + s; 2u, 2v, 4g, 4g) \forall s \in \mathbb{N} \).

(ii) Apply map (4) to (i).

\[\Box\]

We note that the case of Theorem 2.1, which generates full orthogonal designs, was studied in [7]. The application of Theorem 2.1 is illustrated with the following example.

**Example 2.2.** We consider the \( 2 - \text{NPAF}(3; 1, 4) \) taken from [6]:

\[
\begin{align*}
A &= [a, b, -a] \\
B &= [a, 0, a]
\end{align*}
\]

By map (9) there are \( \text{GS}(2) \rightarrow 2 - \text{NPAF}(4; 4, 4) \), as follows:

\[
P' = [x, x, y, -y] \quad \text{and} \quad Q' = [-x, x, -y, -y].
\]

Then by Theorem 2.1 the sequences \( P, Q, R, S \) are \( 2 - \text{NPAF}(3; 1, 4) \times \text{GS}(2) \rightarrow 4 - \text{NPAF}(7; 2, 8, 8, 8) \):

\[
\begin{align*}
P &= [a, b, -a, x, x, y, -y] \\
Q &= [a, b, -a, -x, -x, -y, y] \\
R &= [a, 0, a, -x, x, -y, -y] \\
S &= [a, 0, a, x, -x, y, y]
\end{align*}
\]

Using maps (7) and (4) we obtain: \( 4 - \text{NPAF}(7; 2, 8, 8, 8) \rightarrow 4 - \text{NPAF}(7 + s; 2, 8, 8, 8) \rightarrow \text{OD}(28 + 4s; 2, 8, 8, 8) \forall s \in \mathbb{N}. \)

\[\Box\]
2.2 Orthogonal designs from directed sequences

Theorem 2.1, has some Corollaries pertaining directed sequences (DS). We will say that some sequences of variables are **directed** if the sequences have zero autocorrelation function independently from the properties of the variables, i.e. they do not rely on commutativity to ensure zero NPAF. Clearly, two directed sequences are $2-NPAF(n; u, u)$ and will be denoted by $DS(n; u, u)$. Directed sequences were introduced in [6] and have been used extensively so far to construct orthogonal designs. In addition, orthogonal designs constructed by directed sequences will be called directed orthogonal designs.

**Corollary 2.3.** Suppose there exist $DS(m; n, n)$ and $GS(g)$. Then there exist the following maps (constructions) for $DS$ and orthogonal designs:

(i) $DS(m; n, n) \times GS(g) \rightarrow 4-NPAF(m + 2g + s; 2n, 2n, 4g, 4g) \forall s \in \mathbb{IN}$

(ii) $DS(m; n, n) \times GS(g) \rightarrow OD(4 \cdot (m + 2g + s); 2n, 2n, 4g, 4g) \forall s \in \mathbb{IN}$

**Proof.** (i) $DS(m; n, n)$ are $2-NPAF(m; n, n)$ and from Theorem 2.1 we obtain the following map: $2-NPAF(m; n, n) \times GS(g) \rightarrow 4-NPAF(m + 2g + s; 2n, 2n, 4g, 4g) \forall s \in \mathbb{IN}$. In addition, we need to show that the $4-NPAF(m + 2g + s; 2n, 2n, 4g, 4g)$ sequences are directed. Since the initial sequences are directed we have that in the construction of Theorem 2.1 the terms arising in the NPAF of $P$ from $P'$ will be negated in the NPAF of $Q$ from $-P'$ without relying on commutativity. The same holds for $Q'$ in the sequences $R$ and $S$. Therefore, the sequences $P, Q, R, S$ with zero NPAF, are directed.

(ii) Apply map (4) to (i).

**Remark 2.4.** The converse of Corollary 2.3 (which can be regarded as a ramification of Theorem 2.1) is not true. In Example 2.2 the sequences $P, Q, R, S$ with zero NPAF, are not directed ($P_P(1) + P_Q(1) + P_R(1) + P_S(1) = 2(ab - ba) = 0$ if and only if $a$ and $b$ are commuting variables). This occurs, since the sequences $A = [a, b, -a]$ and $B = [a, 0, a]$ with zero NPAF, are not directed ($N_A(1) + N_B(1) = ab - ba = 0$ if and only if $a$ and $b$ are commuting variables).

The main advantage of directed sequences, their multiplication property, is that their variables can be replaced by sequences with zero NPAF to obtain longer sequences of different type, with zero NPAF, suitable for the construction of large orthogonal designs. This is illustrated in the following example.
Example 2.5. There are $2 - NPAF(4; 4, 4)$ and $GS(2)$. Then from Theorem 2.1, there exist $2 - NPAF(4; 4, 4) \times GS(2) \rightarrow 4 - NPAF(8; 8, 8, 8, 8)$ sequences:

\[
P = \begin{bmatrix} a, & a, & b, & -b, & c, & c, & d, & -d \end{bmatrix} \\
Q = \begin{bmatrix} a, & a, & b, & -b, & -c, & -c, & -d, & d \end{bmatrix} \\
R = \begin{bmatrix} a, & -a, & b, & b, & -c, & c, & -d, & d \end{bmatrix} \\
S = \begin{bmatrix} a, & -a, & b, & b, & c, & -c, & d, & d \end{bmatrix}
\]

The following sequences are $DS(4; 4, 4)$:

\[
A = [a, a, b, -b] \quad \text{and} \quad B = [-a, b, b].
\]

Therefore we can replace the variables of the sequences $A$ and $B$ by the following $2 - NPAF(2; 2, 2)$ sequences $F$ and $G$ taken from [6] to obtain longer sequences with zero NPAF.

\[
F = [e, f] \quad \text{and} \quad G = [e, -f].
\]

In particular, by replacing the variables $a, b$ in the sequences $A, B$ with the sequences $F, G$ respectively we obtain a map $DS(4; 4, 4) \times 2 - NPAF(2; 2, 2) \rightarrow DS(8; 8, 8, 8)$:

\[
A' = [e, f, e, f, e, -f, -e, f] \quad \text{and} \quad B' = [e, f, -e, -f, e, -f, e, -f].
\]

$A', B'$, are then plugged-into the sequences $P, Q, R, S$ to obtain a new set of $4 - NPAF(12; 8, 8, 16, 16)$ sequences $P', Q', R', S'$:

\[
P' = \begin{bmatrix} e, & f, & e, & f, & e, & -f, & -e, & f, & c, & c, & d, & -d \end{bmatrix} \\
Q' = \begin{bmatrix} e, & f, & e, & f, & e, & -f, & -e, & f, & -c, & -c, & -d, & d \end{bmatrix} \\
R' = \begin{bmatrix} e, & f, & -e, & -f, & e, & -f, & e, & -f, & c, & c, & -d, & -d \end{bmatrix} \\
S' = \begin{bmatrix} e, & f, & -e, & -f, & e, & -f, & e, & -f, & c, & -c, & d, & d \end{bmatrix}
\]

and by applying maps (7) and (4) we obtain: $4 - NPAF(12; 8, 8, 16, 16) \rightarrow 4 - NPAF(12 + s; 8, 8, 16, 16) \rightarrow OD(48 + 4s; 8, 8, 16, 16) \forall s \in \mathbb{N}$. □

Corollary 2.6. Suppose there exist $DS(m; n, n), GS(g)$ and $2 - NPAF(k; u, v)$. Then there exist the following maps (constructions) for $DS$ and orthogonal designs $\forall s \in \mathbb{N}$:

(i) $DS(m; n, n) \times 2 - NPAF(k; u, v) \times GS(g) \rightarrow 4 - NPAF(mk + 2g + s; 2nu, 2nv, 4g, 4g)$

(ii) $DS(m; n, n)\times 2 - NPAF(k; u, v)\times GS(g) \rightarrow OD(4\cdot(mk+2g+s); 2nu, 2nv, 4g, 4g)$
Proof. (i) Mimicking the proof of Corollary 2.3 we obtain a map $DS(m; n, n) \times 2 - NPAF(k; u, v) \to DS(mk; nu, nv)$. The derived sequences are $2 - NPAF(mk; nu, nv)$, and can be used in Theorem 2.1 we obtain the desired result.

(ii) Use map (4) to (i).

Set $M = \{(2, 2, 2), (6, 5, 5), (10, 10, 10), (14, 13, 13), (24, 17, 17), (26, 26, 26), (30, 25, 25), (40, 34, 34)\}$ and $N = \{(3, 1, 4), (6, 2, 8), (6, 5, 5), (10, 10, 10), (10, 4, 16), (14, 13, 13), (18, 5, 20), (20, 8, 32), (24, 17, 17), (26, 26, 26), (30, 10, 40), (30, 25, 25), (40, 34, 34), (42, 13, 52)\}$.

**Corollary 2.7.** Let $x_1 \geq 0, x_2 \geq 0, \ldots, x_8 \geq 0, s \geq 0$ be integer numbers. There exist an orthogonal design $OD(4(k \cdot 2^{x_1} \cdot 6^{x_2} \cdot 10^{x_3} \cdot 14^{x_4} \cdot 24^{x_5} \cdot 26^{x_6} \cdot 30^{x_7} \cdot 40^{x_8} + 2g + s); u \cdot 2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8}, v \cdot 2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8}, 4g, 4g)$ for all $(k, u, v) \in N$.

**Proof.** There are $DS(s; t, t)$ for all $(s, t, t) \in M$ (see [6]). For any integers $x_1 \geq 0, x_2 \geq 0, \ldots, x_8 \geq 0$ one can construct $DS(2^{x_1} \cdot 6^{x_2} \cdot 10^{x_3} \cdot 14^{x_4} \cdot 24^{x_5} \cdot 26^{x_6} \cdot 30^{x_7} \cdot 40^{x_8}, 2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8}, 2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8})$. These sequences are plugged in the four sequences of Theorem 2.1 to give the desirable sequences with zero NPAF. Then, for $(k, u, v) \in N$, we use the $2 - NPAF(k; u, v)$ sequences from [6] and by applying map (7) the result follows.

**Example 2.8.** For $x_2 = 1$, and $x_5 = 1$, whereas $x_i = 0, i = 1, 3, 4, 6, 7, 8$ we obtain $DS(144; 85, 85)$. There exist $2 - NPAF(6; 2, 8)$ and $GS(10)$. From Corollary 2.6 we obtain an infinite family of orthogonal designs: $DS(144; 85, 85) \times 2 - NPAF(6; 2, 8) \times GS(10) \to 4 - NPAF(20776 + s; 40, 40, 170, 680) \to OD(20776 + 4s; 40, 40, 170, 680) \forall s \in \mathbb{N}$.

As it is obvious, the suggested multiplication constructions for orthogonal designs and sequences with zero non-periodic autocorrelation will be useful for the construction of large orthogonal designs. Moreover, it is believed that these methods might be helpful in the theoretical study of orthogonal designs and their asymptotic properties.

### 2.3 Orthogonal designs from Golay sequences

**Theorem 2.9.** Suppose there exist $GS(g_1)$ and $GS(g_2)$. Then there exist the following maps (constructions) for directed CS and orthogonal designs:

(i) $GS(g_1) \times GS(g_2) \to 4 - NPAF(g_1 + 2g_2 + s; 4g_1, 4g_2, 4g_2) \forall s \in \mathbb{N}$
(ii) \( GS(g_1) \times GS(g_2) \rightarrow OD(4 \cdot (g_1 + 2g_2 + s); 4g_1, 4g_2, 4g_2) \ \forall \ s \in \mathbb{N} \)

Proof.  
(i) There exists the following map from Theorem 10 of [7] and combining it with map (7) we obtain: \( GS(g_1) \times GS(g_2) \rightarrow 4 - NPAF(g_1 + 2g_2; 4g_1, 4g_2, 4g_2) \rightarrow 4 - NPAF(g_1 + 2g_2 + s; 4g_1, 4g_2, 4g_2). \) In addition, map (9) ensures the intermediate sequences are directed. The other terms of the NPAF of the resulting sequences are zero without relying on the commutativity of the involved variables.

(ii) Apply map (4) to (i).

We note that the case of Theorem 2.9, which generates full orthogonal designs, was studied in [7]. The application of Theorem 2.9 is illustrated with the following example.

Example 2.10. There are \( GS(10) \) from [3], and we apply map (8) \( GS(10) \rightarrow 2 - NPAF(10; 20) \). By map (9) there are \( GS(2) \rightarrow 2 - NPAF(4; 4, 4) \), as follows: 

\[
P' = [x, x, y, -y] \quad \text{and} \quad Q' = [-x, x, -y, -y].
\]

Then by Theorem 2.9 we construct \( GS(10) \times GS(2) \rightarrow 4 - NPAF(14; 8, 8, 40). \) Using maps (7) and (4) we obtain: \( 4 - NPAF(14; 8, 8, 40) \rightarrow 4 - NPAF(14 + s; 8, 8, 40) \rightarrow OD(56 + 4s; 8, 8, 40) \ \forall \ s \in \mathbb{N}. \)

Golay sequences are used in applied areas such as design of optical instruments and communication engineering, as well as in combinatorics for construction of orthogonal designs and Hadamard matrices. We reformulate below our construction for orthogonal designs using Golay sequences, by applying the previous composition construction.

Corollary 2.11. There exists an infinite family of three variable directed orthogonal designs, \( OD(4 \cdot (2^a \cdot 10^b \cdot 26^c + 2^{e+1} \cdot 10^f \cdot 26^g + s); 2^{a+2} \cdot 10^b \cdot 26^c, 2^{e+2} \cdot 10^f \cdot 26^g, 2^{e+2} \cdot 10^f \cdot 26^g) \) for all \( s \geq 0 \) where \( a, b, c, d, e, f \) nonnegative integers.

Proof. We can construct \( A, B \) and \( G, H \) Golay sequences of lengths \( n_1 = 2^a \cdot 10^b \cdot 26^c, n_2 = 2^e \cdot 10^f \cdot 26^g \), respectively, where \( a, b, c, d, e, f \) nonnegative integers from map (10). By plugging these sequences in the construction of Theorem 2.9 we obtain the desired result. In particular, the sequences \( P' = [Gx \ | \ Hy] \) and \( Q' = [H^*x \ | \ -G^*y] \) are \( 2 - NPAF(2^{e+1} \cdot 10^f \cdot 26^g, 2^{e+1} \cdot 10^f \cdot 26^g, 2^{e+1} \cdot 10^f \cdot 26^g). \) This GS pair are doubled doubled in the construction of Theorem 2.9:

\[
P = [ A \ | \ P' ]
Q = [ A \ | \ -P' ]
R = [ B \ | \ Q' ]
S = [ B \ | \ -Q' ]
\]

and the result follows. \( \square \)
3 New weighing matrices from complementary sequences

Orthogonal designs with fewer variables can be obtained by Equating and Killing variables \([2]\) from the orthogonal designs constructed from complementary sequences and produce weighing matrices, since an orthogonal design \(OD(n; k)\) is equivalent to a weighing matrix \(W = W(n, k)\). In a similar way, orthogonal designs derived by complementary sequences (i.e. sequences with zero non-periodic autocorrelation function) can produce infinite families of weighing matrices.

3.1 An infinite family of \(W(144 + 4s, 144)\)

**Corollary 3.1.** There exists an infinite family of orthogonal designs, \(OD(144 + 4s; 32, 32, 80)\). Furthermore, there exists an infinite family of weighing matrices \(W(144 + 4s, 144), \forall s \in \mathbb{N}\).

**Proof.** There exist \(GS(20)\) and \(GS(8)\) from map \((10)\), \(GS(2) \times GS(10) \rightarrow GS(20)\) and \((GS(2) \times GS(2)) \times GS(2) \rightarrow GS(8)\).

From Theorem 2.9 we obtain \(GS(20) \times GS(8) \rightarrow 4 - NPAF(36 + s; 32, 32, 80)\). Using map \((4)\) we obtain \(OD(144 + 4s; 32, 32, 80), \forall s \in \mathbb{N}\). By applying map \((5)\) we obtain \(4 - NPAF(36; 32, 32, 80) \rightarrow 4 - CS(36, 144):\)

```
+++-----------------------------------
+++++----------------------------------+
+++++++---------------------------------+
+++++----------------------------------+
```

where combining maps \((6)\) and \((3)\) we obtain: \(4 - CS(36, 144) \rightarrow 4 - CS(36 + s, 144) \rightarrow W(144 + 4s, 144), \forall s \in \mathbb{N}\). □

3.2 An infinite family of \(W(224 + 4s, 196)\)

**Corollary 3.2.** There exists an infinite family of orthogonal designs, \(OD(224 + 4s; 34, 34, 64, 64)\). Furthermore, there exists an infinite family of weighing matrices \(W(224 + 4s, 196), \forall s \in \mathbb{N}\).

**Proof.** There exist \(2 - NPAF(24; 17, 17)\) from \([6]\) and \(GS(16)\) from map \((10)\), \(((GS(2) \times GS(2)) \times GS(2)) \times GS(2) \rightarrow GS(16)\).

From Theorem 2.1 we obtain \(2 - NPAF(24; 17, 17) \times GS(16) \rightarrow 4 - NPAF(56 + s; 34, 34, 64, 64)\). Using map \((4)\) we obtain \(OD(224 + 4s; 34, 34, 64, 64), \forall s \in \mathbb{N}\). By applying map \((5)\) we obtain \(4 - NPAF(56; 34, 34, 64, 64) \rightarrow 4 - CS(56, 196):\)
where combining maps (6) and (3) we obtain: $4 - CS(56, 196) \rightarrow 4 - CS(56 + s, 196) \rightarrow W(224 + 4s, 196), \forall s \in \mathbb{N}$.

3.3 The new weighing matrices

Weighing matrices $W(n_1, 144)$ for $n_1 = 148, 152, 156, 164, 172, 176$ and $W(n_2, 196)$ for $n_2 = 224, 232, 236, 244, 248, 252, 260, 264, 268, 276, 280, 284, 292, 296, 308, 316$ are listed as open in the second edition of the Handbook of Combinatorial Designs, [1]. The two families $W(144 + 4s, 144)$ and $W(224 + 4s, 196)$ thus give a construction for each one of the previous 22 cases of weighing matrices. However, in a recent paper, [8], two families $W(160 + 4s, 144)$ and $W(200 + 4s, 196)$ of weighing matrices were given by a different method using complementary sequences derived by near normal sequences (for details and undefined terms, also see [8]). Therefore, the $W(148, 144), W(152, 144), W(156, 144)$ constructed in this Section are entirely new.

References


Received: April, 2010