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Abstract

This paper develops approximate analytical solutions of the general form of Lotka–Volterra, Prey–Predator system using the perturbation technique. Here it is shown that results of the previous work [3] become a particular case of the present work.

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1. Introduction

The mathematical study of Prey–Predator systems in population dynamics has been the subject of several recent papers starting with the work of Lotka–Volterra. In the study of non linear system of differential equations such as in the Lotka–Volterra equation, analytical solutions are usually unknown. In this case,
in order to analyze the behavior of the system, we usually resort to numerically integrated techniques such as perturbation technique. The perturbation technique depends on the existence of small or large parameters in the non-linear problems.

The general form of equations which we considered here is

\[\dot{x}_1 = \left(a_1 - b_1 x_1 - c_1 x_2\right) x_1\]
\[\dot{x}_2 = \left(-a_2 + b_2 x_1 - c_2 x_2\right) x_2\]

Here \(x_1\) is prey population and \(x_2\) is predator population where \(a_1, b_1, c_1, a_2, b_2, c_2\) are all positive constants with negative feedback i.e., \(\frac{a_1}{a_2} > \frac{b_1}{b_2}\).


\[\dot{x}_1 = \left(a_1 - c_1 x_2\right) x_1\]
\[\dot{x}_2 = \left(-a_2 + c_2 x_1\right) x_2\]

under the assumption \(a_1 = a_2\).

Wilson [6] gave the form of exact solutions of (1.2) with the assumption namely,

\[a_1(t) = a_2(t), \quad b_1(t) = kb_2(t)\] where

\(a_1, a_2, b_1, b_2\) are functions of time \(t\), where ‘\(k\)’ is a constant. Burnside [1] gave the form of exact solutions with an assumption,

\[(a_1 - a_2)c_1c_2 = c_1 \dot{c}_1 - c_1 \dot{c}_2\]


\[\dot{x}_1 = \left(a_1 - b_1 x_1 - c_1 x_2\right) x_1\]
\[\dot{x}_2 = \left(-a_2 + b_2 x_1\right) x_2\]

This paper develops approximate analytical solutions of the general Lotka–Volterra equations (1.1) using perturbation technique. Here it is shown that the solutions of (1.3) become a particular case of (1.1).

2. Perturbation Method

The perturbation method [2] has been used widely in non-linear mechanics. This method can be applied to a pair of first order differential equations of the type.

\[\dot{x}(t) = f_1(x, y, t) + \mu g_1(x, y, t)\]
\[\dot{y}(t) = f_2(x, y, t) + \mu g_2(x, y, t)\]

with the initial conditions \(x(0) = P_0\) and \(y(0) = P_1\). The linear terms are written in functions \(f_1\) and \(f_2\). The non-linear terms are written in functions \(g_1\) and \(g_2\). The parameter \(\mu\), ideally a small parameter is associated with \(g_1\) and \(g_2\). The solution is found as a power series in \(\mu\). This series will converge rapidly if \(\mu\) is small. In practice \(\mu\) is introduced artificially. The series solution sought is of the form
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\[ x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \ldots \quad (2.2) \]
\[ y(t) = y_0(t) + \mu y_1(t) + \mu^2 y_2(t) + \ldots \]

The terms \( x_0(t) \) and \( y_0(t) \) are called generating terms and are the exact solutions of the linear equations
\[ \dot{x}(t) = f_1(x, y, t) \]
\[ \dot{y}(t) = f_2(x, y, t). \]

The terms \( x_1(t), y_1(t), x_2(t), y_2(t) \) etc... are called correction terms. An important feature of the perturbation method is that both generating and correction terms in the solutions are obtained by solving linear differential equations only.

3. Application to general prey – predator system of equations

The Lotka – Volterra equations (1.1) have the equilibrium point at
\[ \alpha = \frac{a_1c_2 + a_2c_1}{b_1c_2 + b_2c_1}, \quad \beta = \frac{a_1b_2 - a_2b_1}{b_1c_2 + b_2c_1} \quad (3.1) \]

besides the equilibrium points \((0, 0), \left( \frac{a_1}{b_1}, 0 \right), \left( 0, \frac{-a_2}{c_2} \right)\)

Defining the variables \( y_1 = x_1 - \alpha, \quad y_2 = x_2 - \beta \),
we can obtain the differential equations in \( y_1 \) and \( y_2 \) as given below. The variables \( y_1 \) and \( y_2 \) represent prey and predator population deviations from the equilibrium value, given in equation (3.1)
\[ \dot{y}_1 = -b_1 \alpha y_1 - c_1 \alpha y_2 - c_1 y_1 y_2 - b_3 y_1^2 \]
\[ \dot{y}_2 = b_2 \beta y_1 - c_2 \beta y_2 + b_2 y_1 y_2 - c_2 y_2^2 \quad (3.3) \]

Introducing the parameter \( \mu \) into the nonlinear terms of the above equations by applying the perturbation technique.
\[ \dot{y}_1 = -b_1 \alpha y_1 - c_1 \alpha y_2 - c_1 \mu y_1 y_2 - b_3 \mu y_1^2 \]
\[ \dot{y}_2 = b_2 \beta y_1 - c_2 \beta y_2 + b_2 \mu y_1 y_2 - c_2 \mu y_2^2 \quad (3.4) \]

Solutions are sought in the form
\[ y_1(t) = y_{10}(t) + \mu y_{11}(t) + \mu^2 y_{12}(t) + \ldots \quad (3.5) \]
\[ y_2(t) = y_{20}(t) + \mu y_{21}(t) + \mu^2 y_{22}(t) + \ldots \quad (3.5) \]

The parameter \( \mu \) will be set equal to unity after the solutions are determined. By substituting (3.5) into (3.4) and equating equal powers of \( \mu \) on both sides, we obtain
\[ \dot{y}_{10} = -b_1 \alpha y_{10} - c_1 \alpha y_{20} \quad (3.6) \]
\[ \dot{y}_{20} = -c_2 \beta y_{20} + b_2 \beta y_{10} \]
\[ \dot{y}_{11} = -b_1 y_{10}^2 - b_1 \alpha y_{11} - c_1 \alpha y_{21} - c_1 y_{10} y_{20} \]
\[ \dot{y}_{21} = -c_2 y_{20}^2 - c_2 \beta y_{21} + b_2 \beta y_{11} + b_2 y_{10} y_{20} \quad (3.7) \]
The initial conditions \( y_{10}(0) = p_0, \ y_{20}(0) = p_1 \) are applied to the generating solutions \( y_{10}(t) \) and \( y_{20}(t) \). The differential equations for corresponding terms then have zero initial conditions. The solutions of the linear equations (3.6) are obtained as

\[
y_{10}(t) = \frac{1}{k_1 - k_2} \left[ (k_1 e^{k_1 t} - k_2 e^{k_2 t}) p_0 + (p_0 c_2 \beta - p_1 c_1 \alpha) (e^{k_1 t} - e^{k_2 t}) \right] \tag{3.8}
\]

\[
y_{20}(t) = \frac{1}{k_1 - k_2} \left[ (k_1 e^{k_1 t} - k_2 e^{k_2 t}) p_1 + (p_0 b_2 \beta + p_1 b_1 \alpha) (e^{k_1 t} - e^{k_2 t}) \right] \tag{3.9}
\]

where

\[
k_1 = -\frac{(ab_1 + c_2 \beta) + \sqrt{(ab_1 + c_2 \beta)^2 - 4 \alpha \beta (c_1 b_2 + c_2 b_1)}}{2}
\]

\[
k_2 = -\frac{(ab_1 + c_2 \beta) - \sqrt{(ab_1 + c_2 \beta)^2 - 4 \alpha \beta (c_1 b_2 + c_2 b_1)}}{2}
\]

where \( \alpha = \frac{a c_2 - a_2 c_1}{b_1 c_2 + b_2 c_1} \), \( \beta = \frac{a_1 b_2 - a_2 b_1}{b_1 c_2 + b_2 c_1} \) with \( k_1 - k_2 \neq 0 \)

where \( p_0 \) and \( p_1 \) are initial conditions of \( y_{10}(0) \) and \( y_{20}(0) \) respectively. Having determined \( y_{10}(t) \) and \( y_{20}(t) \) equations (3.7) we next solved for \( y_{11}(t) \) and \( y_{21}(t) \) to yield.

\[
y_{11}(t) = A_1 e^{k_1 t} + A_2 e^{k_2 t} + A_3 e^{2k_1 t} + A_4 e^{2k_2 t} + A_5 e^{(k_1 + k_2)t} \tag{3.10}
\]

\[
y_{21}(t) = B_1 e^{k_1 t} + B_2 e^{k_2 t} + B_3 e^{2k_1 t} + B_4 e^{2k_2 t} + B_5 e^{(k_1 + k_2)t}
\]

where

\[
A_1 = \frac{1}{(k_1 - k_2)^3} \left[ \frac{1}{2k_2 - k_1} \left\{ (2k_2 + \alpha c_1 c_2)(c_1 M + b_1 Q) + (b_2 M - c_2 F)b_1 c_1 \alpha \right\} - \frac{1}{k_2} \left\{ (k_1 + k_2 + \alpha c_1 c_2)(c_1 N + 2b_1 R) + (b_2 N - 2c_2 G)b_1 c_1 \alpha \right\} + \frac{1}{k_1} \left\{ (2k_1 + \alpha c_1 c_2)(c_1 L + b_1 P) + (b_2 L - c_2 E)b_1 c_1 \alpha \right\} \right]
\]

\[
A_2 = \frac{1}{(k_1 - k_2)^3} \left[ -\frac{1}{2k_1 - k_2} \left\{ (2k_1 + \alpha c_1 c_2)(c_1 L + b_1 P) + (b_2 L - c_2 E)b_1 c_1 \alpha \right\} + \frac{1}{k_1} \left\{ (k_1 + k_2 + \alpha c_1 c_2)(c_1 N + 2b_1 R) + (b_2 N - 2c_2 G)b_1 c_1 \alpha \right\} \right]
\]
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\[-\frac{1}{k_2} \left\{ (2k_2 + \alpha c_1 c_2) (c_1 M + b_i Q) + (b_2 M - c_2 F) b_i c_1 \alpha \right\} \]

\[A_3 = \frac{1}{(k_1 - k_2)^2} \left[ - \frac{1}{k_1 (2k_1 - k_2)} \left\{ (2k_1 + \alpha c_1 c_2) (c_1 L + b_i P) + (b_1 L - c_1 E) b_i c_1 \alpha \right\} \right] \]

\[A_4 = \frac{1}{(k_1 - k_2)^2} \left[ - \frac{1}{k_1 (2k_2 - k_1)} \left\{ (2k_2 + \alpha c_1 c_2) (c_1 M + b_i Q) + (b_2 M - c_2 F) b_i c_1 \alpha \right\} \right] \]

\[A_5 = \frac{1}{(k_1 - k_2)^2} \left[ \frac{1}{k_1 k_2} \left\{ (k_1 + k_2 + \alpha c_1 c_2) (2b_i R + c_1 N) + (b_2 N - 2c_2 G) b_i c_1 \alpha \right\} \right] \]

\[B_1 = \frac{-1}{(k_1 - k_2)^3} \left[ \frac{1}{2k_2 - k_1} \left\{ (b_i^2 \alpha + 2k_i) (b_2 L - c_2 E) - (c_i M + b_i Q) b_2 c_1 \alpha \right\} \right] \]

\[B_2 = \left\{ b_i^2 \alpha + k_1 + k_2 \right\} \frac{1}{2k_2 - k_1} \left\{ (2c_2 G - b_i N) + (c_1 N + 2b_i R) b_2 c_1 \alpha \right\} \]

\[B_3 = \frac{1}{(k_1 - k_2)^2} \left[ \frac{1}{k_1 (2k_1 - k_2)} \left\{ (b_i^2 \alpha + 2k_i) (b_2 L - c_2 E) - (c_i L + b_i P) b_2 c_1 \alpha \right\} \right] \]

\[B_4 = \frac{1}{(k_1 - k_2)^2} \left[ \frac{1}{k_1 (2k_2 - k_1)} \left\{ (b_i^2 \alpha + 2k_i) (b_2 M - c_2 F) - (c_i M + b_i Q) b_2 c_1 \alpha \right\} \right] \]

\[B_5 = \frac{1}{(k_1 - k_2)^2} \left[ \frac{1}{k_1 k_2} \left\{ (b_i^2 \alpha + k_1 + k_2) (2c_2 G - b_i N) + (2b_i R + c_1 N) b_2 c_1 \alpha \right\} \right] \]

and

\[P = ((k_1 + \beta c_2) p_0 - p_i c_i \alpha) \]

\[Q = (k_2 + \beta c_2) p_0 - p_i c_i \alpha) \]

\[R = ((k_1 + \beta c_2) p_0 - p_i c_i \alpha) ((k_2 + \beta c_2) p_0 - p_i c_i \alpha) \]

\[L = ((k_1 + \beta c_2) p_0 - p_i c_i \alpha) ((k_2 + \beta c_2) p_0 - p_i c_i \alpha) \]
Using these first correction terms, an appropriate solution to (3.3) is
\[
y_1(t) \cong y_{10} + y_{11}(t)
\]
\[
y_2(t) \cong y_{20} + y_{21}(t)
\]

In terms of the original prey and predator populations \(x_1(t)\) and \(x_2(t)\), the solutions are
\[
x_1(t) = \frac{a_1c_1 + a_2c_1}{b_1c_2 + b_2c_1} + \left[\frac{(k_1 + \beta c_2 - p_1c_1\alpha)(k_2 + \alpha b_1) + p_1 + p_0b_2\beta}{k_1 - k_2} + A_1\right]e^{k_1t} - \left[\frac{(k_1 + \beta c_2 - p_1c_1\alpha)(k_2 + \alpha b_1) + p_1 + p_0b_2\beta}{k_1 - k_2} - A_2\right]e^{k_2t} + A_3e^{2k_1t} + A_4e^{2k_2t} + A_5e^{(k_1+k_2)t} \tag{3.12a}
\]
\[
x_2(t) = \frac{a_2b_1 - a_1b_1}{b_1c_2 + b_2c_1} + \left[\frac{(k_1 + \alpha b_1)p_1 + p_0b_2\beta}{k_1 - k_2} + B_1\right]e^{k_1t} - \left[\frac{(k_1 + \alpha b_1)p_1 + p_0b_2\beta}{k_1 - k_2} - B_2\right]e^{k_2t} + B_3e^{2k_1t} + B_4e^{2k_2t} + B_5e^{(k_1+k_2)t} \tag{3.12b}
\]
Substituting \(c_2 = 0\) in (3.12a), (3.12b), the equilibrium point of (1.3) is
\[
\begin{bmatrix}
a_2 \\
b_2 \\
c_1b_2 \\
a_2b_2 - a_1b_1 \\
b_1 \\
c_1
\end{bmatrix}, \quad k_i = -\alpha + \sqrt{\alpha^2 - 4\gamma \beta}, \quad k_2 = -\frac{\alpha - \sqrt{\alpha^2 - 4\gamma \beta}}{2}
\]
with \(\alpha = \frac{b_1a_2}{b_2}, \beta = \frac{c_1a_2}{b_2}, \gamma = \frac{a_2b_2 - a_1b_1}{c_1}\)

Then we get approximate analytical solutions of (1.3) of the form
\[
x_1(t) = \frac{a_2}{b_2} + \left[\frac{k_1p_0 - p_1c_1\alpha}{k_1 - k_2} + A_1\right]e^{k_1t} - \left[\frac{k_2p_0 - p_1c_1\alpha}{k_1 - k_2} - A_2\right]e^{k_2t} + A_3e^{2k_1t} + A_4e^{2k_2t} + A_5e^{(k_1+k_2)t}
\]
\[
x_2(t) = \frac{a_1b_1 - a_2b_1}{c_1b_2} + \left[\frac{(k_1 + \alpha)p_1 + p_0\gamma}{k_1 - k_2} + B_1\right]e^{k_1t} - \left[\frac{(k_1 + \alpha)p_1 + p_0\gamma}{k_1 - k_2} - B_2\right]e^{k_2t} + B_3e^{2k_1t} + B_4e^{2k_2t} + B_5e^{(k_1+k_2)t}
\]
where \(A_i, B_i\) constants obtained by taking \(c_2 = 0\) in (3.12a) and (3.12b).
Here it is shown that the solutions of the Lotka–volterra Equations (1.3) became a particular case of the general Lotka–Volterra Equations (1.1). Because of the additional correction terms, the above solutions can be expected to be more accurate and be valid for larger deviations from the equilibrium point. The accuracy of the solutions by perturbation method can be improved further by determine second and higher order correction terms.

4. Conclusions

The approximate analytical solutions of the general Lotka–Volterra Equations have been determined using perturbation method. The perturbation method can be applied to other non liner mathematical models of population dynamics.

References


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