On the Constants $C(\Omega)$ and $C_s(\Omega)$ of a C*-algebra and Norms of Derivations

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Abstract
We investigate the relationship between inner derivations implemented by a norm-attainable element of a C*-algebra to those of ideals and primitive ideals. Moreover, we give related results on the relationship between the constants $C(\Omega)$ and $C_s(\Omega)$ of C*-algebras to those of ideals and primitive ideals.

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1 Introduction
Let $H$ be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. For a fixed $A \in B(H)$, the inner derivation induced by $A$ is
the operator $\delta_A$ defined on $B(H)$ by $\delta_A(X) = AX -XA, \forall X \in B(H)$. The norm of inner derivation $\delta_A$ on $B(H)$ has been computed by Stampfli [11] as;

$$\| \delta_A \|_{B(H)} = 2d(A)$$

(1)

where $d(A) = \inf\{\| A - \lambda \|; \lambda \in C\}$.

The equality in (1) has received considerable attention mainly in showing that the equality holds in various cases. For example, Kadison, Lance and Ringrose [6], Gajendragadkar [4] showed that the equality holds for all elements of Von-Neumann algebra (not necessarily when they are self adjoint). Stampfli [11] showed that (1) holds when $B(H)$ is a primitive $C^*$-algebra with an identity and in particular when $B(H)$ is the algebra of bounded linear operators in a Banach space. Johnson [5] and Kyle [7] showed that the equality holds and sometimes does not. For quotients of $W^*$-algebras, Sommerset [10] showed that (1) holds while recently, Bonyo and Agure [3] showed that when $J$ is a proper two-sided ideal then (1) is true.

In order to examine the behaviour of norms of these norm-attainable inner derivations, we shall determine the relationship between two constants due to Archbold [2] i.e. $C(\Omega)$ and $C_s(\Omega)$ which he defined to be the smallest number in $[0, \infty]$ such that $d(A, Z(\Omega)) \leq C \| \delta_A \|_\Omega, \forall A \in \Omega$ and $d(A, Z(\Omega)) \leq C_s \| \delta_A \|_\Omega, \forall A \in B(H)$ and $A = A^*$.

We have shown that $C(\Omega) = C_s(\Omega) = 0$ when $\Omega$ is commutative, but when it is non commutative, then from (1), $C(\Omega) \geq \frac{1}{2}$ and $C_s(B(H)) \geq \frac{1}{2}$.

Sommerset [10] established the relationship between $C_s(\Omega)$ and the order of connecting $\Omega$ and $Orc\Omega$. His main result showed that when $\Omega$ is a weakly central $C^*$-algebra, then $Orc\Omega \leq 2$ so that $C_s(\Omega) \leq 1$, hence the relation $C_s(\Omega) = \frac{1}{2}Orc(\Omega)$. In the present paper, we shall investigate the relationship between a norm-attainable inner derivations induced by a norm-attainable element of a $C^*$-algebra algebra to those of ideals and primitive ideals(see details on norms of derivations in [9] and on norm-attainability of derivations in [9]). We have also examined the two constants $C(\Omega)$ and $C_s(\Omega)$ in the interval $[0, \infty]$.

We therefore, organize our work in the following sections: 1. Introduction; 2. Preliminaries; 3. Norms of norm-attainable inner derivations; 4. On the Constants $C(\Omega)$ and $C_s(\Omega)$ of a $C^*$-algebra.

In the next section, we give the basic definitions and the notation we shall use in the sequel.
2 Preliminaries

From this stage and throughout this paper, unless stated otherwise, we take the C*-algebra $B(H) = \Omega$ for convenience.

Definition 2.1. A derivation on a C*-algebra algebra $\Omega$ is a linear transformation $\delta : \Omega \to \Omega$ which satisfies $\delta(AB) = A\delta(B) + B\delta(A)\forall A,B \in \Omega$. If for a fixed $A$, $\delta : X \to AX -XA$, then $\delta$ is an inner derivation.

Definition 2.2. For an operator $A \in \Omega$, the operator $A$ is said to be norm-attainable if there exists a unit vector $x$ such that $\|Ax\| = \|A\|$.

Definition 2.3. A derivation, $\delta_{A,B}$, on a C*-algebra algebra $\Omega$ is said to be norm-attainable if there exists a functional $\varphi \in H^*$ such that $\|\delta_{A,B}\varphi\| = \|\delta_{A,B}\|$ and $\|\varphi\| = 1$.

Remark 2.4. We denote a norm-attainable generalized derivation (inner derivation) by $\delta^N_{A,B}(\delta^N_A)$.

3 Norms of norm-attainable inner derivations

Lemma 3.1. Let $\Omega$ be a C*-algebra, $\text{prim}(\Omega)$ the set of all primitive ideals in $\Omega$, $[A]$ the cannonical image of $A$ in $\Omega/J$, then for a norm-attainable inner derivation $\delta^N_A$, $\| \delta^N_A | \Omega \| = \sup\{ \| \delta^N_A | \Omega/J \| : J \in \text{prim}(\Omega) \}$.

Proof. Clearly, since $A$ is fixed in $\Omega/J$ and from the definition of inner derivation we have, $\| \delta^N_{[A]}([X])\| = \| [A][X] - [X][A] \| = \| [X][A] - [A][X] \|$. Since $\| \delta^N_A | \Omega/J \| = \sup\{ \| \delta^N_{[A]}([X]) : X \in \Omega/J \} \text{ and } \| [X] \| = 1$ it follows that $\| \delta^N_{[A]} | \Omega/J \| \geq \| \delta^N_{[A]}([X]) \|, \forall [X] \in \Omega/J, \| [X] \| = 1$. Also for any $\epsilon > 0$, $\exists [X] \in \Omega/J$ with $\| [X] \| = 1$ such that $\| \delta_{[A]} | \Omega/J \| < \| \delta^N_{[A]}([X]) \| + \epsilon$ and since $\| [X] \| = \inf\{ \| X + K \| : K \in J \}$ we have $\| [X] \| \leq \| X + K \| \forall K \in J$. So again for $\epsilon > 0$ choose $K \in J$ such that $\| X + K \| < \| [X] \| + \epsilon = 1 + \epsilon$. Now we obtain,

$$
\| \delta^N_{[A]} | \Omega/J \| < \| \delta^N_{[A]}([X]) \| + \epsilon \\
= \| [A][X] - [X][A] \| + \epsilon \\
= \| [A(X + K) - (X + K)A] \| + \epsilon \\
= \| \delta^N_A(X + K) \| + \epsilon \\
\leq \| \delta^N_A | \Omega \| \| X + K \| + \epsilon \\
< \| \delta^N_A | \Omega \| (1 + \epsilon) + \epsilon.
$$

Since $\epsilon$ is arbitrary, as $\epsilon \to 0$, we have

$$
\| \delta^N_A | \Omega/J \| \leq \| \delta^N_{[A]} | \Omega \| \tag{2}
$$

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and taking the supremum of (2) we have
\[ \sup\{\| \delta_N^A | \Omega/J \|; J \in \text{prim}(\Omega) \leq \| \delta_N^A | \Omega \| \}. \]

Conversely, for any positive \( \epsilon \), we can find \( X \in \Omega \) with \( \| X \| = 1 \) such that
\[ \| \delta_N^A(X) \| = \| AX - XA \| > \| \delta_N^A | \Omega \| - \epsilon. \]
We also have a primitive ideal \( J \) such that \( \| [AX - XA]_J \| > \| AX - XA \| - \epsilon. \)
But, \( \| [AX - XA]_J \| = \| [A][X] - [X][A] \| \). So \( \| \delta_N^A | \Omega \| - \epsilon < \| [A][X] - [X][A] \| + \epsilon. \)
Taking \( e_\lambda |_{\Lambda} \) as the approximation unit in \( \Omega/J \), we have,
\[
\| [A][X] - [X][A] \| + \epsilon = \lim_{\lambda \in \Lambda} \| [Ae_\lambda][Xe_\lambda] - [Xe_\lambda][Ae_\lambda] \| + \epsilon \\
= \lim_{\lambda \in \Lambda} \| \delta_N^A([Xe_\lambda]) \| + \epsilon \\
\leq \lim \sup_{\lambda \in \Lambda} \| \delta_N^A | J/J \| \| X \| \| e_\lambda \| + \epsilon \\
= \sup_{\lambda \in \Lambda} \| \delta_N^A | J \|; J \in \text{prim}(\Omega) \} + \epsilon.
\]

Since \( \epsilon \) was arbitrary, as \( \epsilon \to 0 \),
\[ \| \delta_N^A | \Omega \| \leq \sup_{\lambda \in \Lambda} \| \delta_N^A | J \|; J \in \text{prim}(\Omega) \} \]
which completes the proof. \( \square \)

**Corollary 3.2.** Let \( A \in \Omega \) be norm-attainable and consider the norm-attainable
inner derivation, \( \delta_N^A \), induced by \( A \). Then the following hold

(i) \( \delta_N^A \) is self-adjoint

(ii) \( A \) is normal

(iii) \( \| \delta_N^A | \Omega \| = 2d(A^*). \)

**Proof.** The proofs of (i) and (ii) are trivial. To prove (iii), since from (i) \( \delta_N^A \)
is self-adjoint, we only need to show that \( \delta_N^A \) is norm-attainable implies that
\( \delta_N^A = (\delta_N^A)^* \) is norm-attainable. If \( \delta_N^A = 0 \), then there is nothing to prove so
let \( \delta_N^A \neq 0 \). If \( \delta_N^A \) is norm-attainable then there is a functional \( \varphi \in H^* \) such
that \( \| \delta_N^A \varphi \| = \| \delta_N^A \| \) and \( \| \varphi \| = 1 \) i.e. \( \| (\delta_N^A)^* \delta_N^A \varphi \| = \| (\delta_N^A)^2 \| \varphi. \)
Let \( \mu_0 = \frac{\delta_N^A \varphi}{\| \delta_N^A \|} \).
Then \( \mu_0 \) is a unit functional and \( \| (\delta_N^A)^* \mu_0 \| = \| (\delta_N^A \| = \| (\delta_N^A)^* \| \).
Indeed,
\[
\| (\delta_N^A)^* \mu_0 \| = \| (\delta_N^A)^* \frac{\delta_N^A \varphi}{\| \delta_N^A \|} \| = \frac{1}{\| \delta_N^A \|} \| (\delta_N^A)^* \delta_N^A \varphi \| = \frac{1}{\| \delta_N^A \|} \| (\delta_N^A)^2 \| \varphi = \| \delta_N^A \|.
\]
But \( \| (\delta_N^A)^* \| = \| \delta_N^A \| \). This completes the proof. \( \square \)
4 The Constants $C'(\Omega)$ and $C'_s(\Omega)$ of a C*-algebra

In this section, we introduce the following constants due to Archbold.

**Definition 4.1.** Let

$$G = \{ C; \ d(A, Z(\Omega)) \leq C \| \delta_A \|, \ A \in \Omega \}$$

and

$$G_s = \{ C; \ d(A, Z(\Omega)) \leq C \| \delta_A \|, \ A \in \Omega (A = A^*) \}.$$  

Denote the greatest lower bound of $G$ by:

$$C(\Omega) = \inf \{ C; \ d(A, Z(\Omega)) \leq C \| \delta_A \|, \ A \in \Omega \}$$

and that of $G_s$ by:

$$C_s = \inf \{ C; \ d(A, Z(\Omega)) \leq C \| \delta_A \|, \ A \in \Omega (A = A^*) \}$$

then $C(\Omega)$ and $C_s(\Omega)$ are called Archbold constants, defined to be the smallest number in $[0, \infty]$.

**Lemma 4.2.** If $\Omega$ is a commutative algebra then $C_s(\Omega) = C(\Omega) = 0$

*Proof.* Clearly,

$$\| \delta_A \| \leq 2d(A, Z(\Omega)), \forall A \in \Omega.$$  

This implies that $\| \delta_A \| = 0$ i.e. $C = 0$. Therefore, $C_s(\Omega) = C(\Omega) = 0$.  

**Lemma 4.3.** Let $\Omega$ be a non-commutative algebra then (i) $C_s(\Omega) = C(\Omega) \geq \frac{1}{2}$ and (ii) $C_s(\Omega) \leq C(\Omega) \leq 2C(\Omega)$.

*Proof.* To prove (i), it is clear that $\| \delta_A \| \leq 2d(A, Z(\Omega)), \forall A \in \Omega$.

So $\frac{1}{2} \| \delta_A \| \leq d(A, Z(\Omega))$. This implies that $d(A, Z(\Omega)) \geq C \| \delta_A \|$, $C \geq \frac{1}{2}$. This completes the proof of part (i).

To prove (ii), consider first $C_s(\Omega) \leq C(\Omega)$. For self adjoint $\Omega$, Stampfli [11] proves that $\| \delta_A \| \leq 2d(A, Z(\Omega))$ holds. So clearly, $C_s(\Omega) \leq \frac{1}{2}$, but generally $\| \delta_A \| \leq 2d(A, Z(\Omega)) \leq 2d(A, Z(\Omega))$ implying that $C(\Omega) \geq \frac{1}{2}$. So it follows immediately that $C_s(\Omega) \leq C(\Omega)$. For the second inequality, let $\forall A \in \Omega$ such that $A = A^*$ let $A = A_i + iA_2, A_i = A_i^*$, $i = 1, 2$ then,
\[ d(A, Z(\Omega)) = d(A_1 + iA_2, Z(\Omega)) \leq d(A_1, Z(\Omega)) + d(A_2, Z(\Omega)) \leq C_s(\Omega) \| \delta A_1 \| + C_s(\Omega) \| \delta A_2 \| \| \Omega \| \leq 2C_s(\Omega) \| \delta A \| \| \Omega \|. \]

From the definition of \( C(\Omega) \), the proof follows immediately. \( \square \)

In the next theorem, we shall consider a \( C^* \)-algebra \( \Omega \) and an ideal \( J \) in \( \Omega \).

**Theorem 4.4.** Let \( C(\Omega) \) be as defined above, and \( C(J) = \inf \{ C; d(A, Z(J)) \leq C \| \delta A \| \| J \|, \forall A \in J \} \), then \( C(\Omega) \leq C(J) \)

**Proof.** Let \( G \) be as defined earlier and \( H = \{ C; d(A, Z(J)) \leq C \| \delta A \| \| J \|, \forall A \in J \} \).

Assuming that \( C(J) < \infty \), it follows that \( C(\Omega) < \infty \) and \( G, H \) are non-empty. So \( C(\Omega) \leq K \) and hence \( C(\Omega) \| \delta A \| \| \Omega \| \leq C \| \delta A \| \| \Omega \| \forall A \in \Omega \). But since \( C(\Omega) = \inf(H) \) for any \( \varepsilon > 0, \exists A \in J \) such that \( C < C(J) + \varepsilon \) hence \( C(\Omega) \| \delta A \| \| \Omega \| < (C/J + \varepsilon) \| \delta A \| \| \Omega \|, \forall A \in \Omega \). So, \( C(\Omega) \| \delta A \| \| \Omega \| \leq C(J) \| \delta A \| \| \Omega \| + \varepsilon \| \delta A \| \| \Omega \|. \) Since \( \varepsilon \) is arbitrary, as \( \varepsilon \to 0 \), from the above inequality we obtain the required result. \( \square \)

**Corollary 4.5.** Let \( \Omega \) be a \( C^* \)-algebra and \( J \in \text{prim}(\Omega) \). Define \( C(\Omega) \) as in the theorem above, and \( C(\Omega/J) = \inf \{ C; d([A], Z(\Omega/J)) \leq C \| \delta [A] \| \| \Omega/J, \forall [A] \in \Omega/J \} \). Then \( C(\Omega) \leq C(\Omega/J) \)

**Proof.** The proof follows from the proof of the above Theorem 4.4. \( \square \)

The following example due to Kyle [7] is useful in demonstrating that \( C(\Omega) \) can also be infinite i.e. \( C(\Omega) = \infty \)

**Example 4.6.**

Let \( \Omega \) be a \( C^\infty(Z^+ \times Z^+, M_2) \), where \( \Omega_2 \) is the algebra of all 2x2 matrices, and let \( \Omega_1 \) be the sub-algebra of \( \Omega \) consisting of all those \((D_{m,n})_{m,n=1}^\infty \) in \( \Omega \) for which \( (D_{m,n})_{1,2} \to 0 \) and \( (D_{m,n})_{2,1} \to 0 \) as \( n \to \infty \) and for which \( \lim_{n \to \infty} (D_{m,n})_{1,1} \), \( \lim_{n \to \infty} (D_{m,n})_{2,2} \) both exist with \( \lim_{n \to \infty} (D_{m,n})_{1,1} \) being equal to \( \lim_{n \to \infty} (D_{m+1,n})_{2,2} \). Indeed, if \( \gamma \) belongs to the centre \( Z(\Omega_1) \) of \( \Omega_1 \), then the rest is clear from [7, Example 3.1].
The constants $C(\Omega)$ and $C_s(\Omega)$ and norms of derivations

References


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