Certain Congruences on Inverse Semirings\textsuperscript{1}

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Abstract

In this paper the maximum additive idempotent separating congruence $\mu$ and minimum skew-ring congruence $\sigma$ on an inverse semiring $S$ are described. The congruence $\sigma \vee \mu$ generated by the two congruences is studied.

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1 Introduction and Preliminaries

A semiring $S$ is an algebraic structure $(S,+,\cdot)$ consisting of a non-empty set $S$ together with two binary operations $+$ and $\cdot$ such that $(S,+)$ and $(S,\cdot)$ are semigroups, connected by ring-like distributivity (that is, $x(y+z) = xy + xz, (y+z)x = yx + zx$, for all $x,y$ and $z$ in $S$). Usually, we write $(S,+,\cdot)$ simply as $S$, and for any $x,y \in S$, write $x \cdot y$ simply as $xy$. A semiring $S$ is a skew-ring if its additive reduct $(S,+)$ is a group. A semiring $S$ is an inverse semiring if $(S,+)$ is an inverse semigroup. A non-empty subset $K$ of a semiring $S$ is a subsemiring if $a+b, ab \in K$ for every $a,b \in K$. A subsemiring $K$ of $S$ is an ideal if $as, sa \in K$ for any $a \in K$ and $s \in S$. A subsemiring $K$ of $S$ is called full(self-conjugate) if its additive reduct $(S,+)$ is full(self-conjugate). A subsemiring of $S$ is called normal if its additive reduct is a full, self-conjugate and inverse subsemigroup of $(S,+)$.

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$\mathcal{C}$ is a class of semirings and $\rho$ is a congruence on a semiring $S$ then $\rho$ is called a $\mathcal{C}$-congruence if $S/\rho \in \mathcal{C}$.

A relation $R$ on a semiring $S$ is called left compatible (right compatible) if
\[
(\forall a, b, c, \in S) \quad (a, b) \in R \Rightarrow (c + a, c + b), (ca, cb) \in R.
\]
\[
((\forall a, b, c, \in S) \quad (a, b) \in R \Rightarrow (a + c, b + c), (ac, bc) \in R)
\]

A relation $\rho$ on $S$ is called a congruence if and only if it is an equivalence and both left and right compatible with the operation on $S$.

If $a$ and $b$ are two elements of an inverse semiring $S$, we write $a \leq b$ if

\[
b + (-b) = b + (-a),
\]

or if any one of the following equivalent conditions holds;

\[
b + (-b) = a + (-b), \quad -b + b = -b + a, \quad -b + b = -a + b.
\]

We can easily verify that the relation $\leq$ is a compatible order relation on the semiring $S$. From some known results of inverse semigroup, we have first,

\[
-a \leq -b \text{ if } a \leq b.
\]

Also, if $e$ is any additive idempotent, $a$ and $b$ are arbitrary elements of $S$, then

\[
a \leq e + a, \quad a \leq a + e, \quad a + b \leq a + e + b.
\]

If $H$ is an arbitrary subset of $S$, we denote by $H\omega$ the closure of $H$ with respect to the above order relation: that is,

\[
H\omega = \{a \in S : a \leq h \text{ for some } h \text{ in } H\}.
\]

Then $H \subseteq H\omega$ for any $H$. A subset $K$ will be called closed if $K\omega = K$. Clearly $H\omega$ is closed for any $H$.

If $E^+(S)$ is the set of additive idempotents of an inverse semiring $S$, we define $E^+\zeta$, the centraliser of $E^+(S)$ in $S$, by

\[
E^+\zeta = \{z \in S : e + z = z + e \text{ for every } e \text{ in } E^+(S)\}.
\]

Clearly $E^+(S) \subseteq E^+\zeta$. If $E^+(S) = E^+\zeta$, we shall say that $E^+(S)$ is self-centralising.

In Howie [1], the structure of inverse semigroup is characterized, some congruences such as maximum idempotent separating congruence and group congruence are studied. Some properties of inverse semigroup are extended to an inverse semiring In [4]. Based on those, the main purpose of this paper is to investigate the maximum additive idempotent separating congruence and minimum skew-ring congruence on an inverse semiring $S$, and give a characterization of the congruence generate by the two congruences. For other notations and terminology about semigroups and semirings not mentioned in this paper, the reader can be referred to [1], [2] and [3].
2 The maximum additive idempotent separating congruence

As a starting point for our investigations we have the following theorem.

**Theorem 2.1.** Let $S$ be an inverse semiring and let $\alpha_a$ be defined by $e\alpha_a = -a + e + a$ for any $e \in E^+(S)$. Then the relation $\mu$ defined by the rule that $(x, y) \in \mu$ if and only if $\alpha_x = \alpha_y$ is the maximum additive idempotent separating congruence on $S$.

**Proof.** It can be easily verified that $\mu$ is an equivalence relation. Now suppose that $(x, y) \in \mu$ and that $z$ is an arbitrary element of $S$. Then from the supposition that $-x + e + x = -y + e + y$ for every additive idempotent $e$, then it follows immediately that $-z + (-x) + e + x + z = -z + (-y) + e + y + z$ for every $e \in E^+(S)$, that is $(x + z, y + z) \in \mu$. Furthermore, we note that $-z + e + z \in E^+(S)$ for every $e \in E^+(S)$, and so $-x + (-z + e + z) + x = -y + (-z + e + z) + y$. Thus $(z + x, z + y) \in \mu$. For any $e \in E^+(S)$, we have $ez, ze \in E^+(S)$, and so

$$-xz + ez + xz = (-x + e + x)z = (-y + e + y)z = -yz + ez + yz \Rightarrow (xz, yz) \in \mu$$

similarly $(zx, zy) \in \mu$. So $\mu$ is a semiring congruence.

Now we show that $\mu$ is additive idempotent separating. Suppose that $(e, f) \in \mu$, where $e$ and $f$ are additive idempotents. Then, for every $g \in E^+(S)$, we have that $-e + g + e = -f + g + f$: that is, $e + g = f + g$. This equality holds in particular when $g = e$; hence $e = f + e$. We similarly obtain that $e + f = f$ by putting $g = f$. Since $e + f = f + e$, it follows that $e = f$. Thus $\mu$ is additive idempotent separating.

Finally, let $\nu$ be an additive idempotent separating congruence on $S$; we shall show that $\nu \subseteq \mu$. Suppose that $(x, y) \in \nu$. Then $(-x, -y) \in \nu$, and, since $\nu$ is a congruence, it follows that $(-x + e + x, -y + e + y) \in \nu$ for every $e \in E^+(S)$. But both $-x + e + x$ and $-y + e + y$ are additive idempotents, and so it follows that $-x + e + x = -y + e + y$. Thus $(x, y) \in \mu$ and so $\nu \subseteq \mu$ as required. This completes the proof of the theorem. \qed

The next theorem gives an alternative characterization of $\mu$.

**Theorem 2.2.** Let $S$ be an inverse semiring, and let $\mu$ be the maximum additive idempotent separating congruence on $S$, $E^+\zeta$ is the centraliser of $E^+(S)$ in $S$. Then $(x, y) \in \mu$ if and only if $-x + x = -y + y$ and $x + (-y) \in E^+\zeta$. Dually, $(x, y) \in \mu$ if and only if $x + (-x) = y + (-y)$ and $(x) + y \in E^+\zeta$. 

Proof. Sufficient. Suppose first that \((x, y) \in \mu\), so that \(-x + e + x = -y + e + y\) for every \(e\) in \(E^+(S)\). Then \((-x, -y) \in \mu\), that is, \(x + e + (-x) = y + e + (-y)\) for every \(e\) in \(E^+(S)\). Hence
\[
-x + x = -x + (x + (-x + x) + (-x)) + x = -x + (y + (-x + x) + (-y)) + x = -y + (y + (-x + x) + (-y)) + y = (-y + y) + (-x + x) + (-y) + y = (-x + x) + (-y) + y.
\]
And similarly \(-y + y = (-x + x) + (-y + y)\). Thus \(-x + x = -y + y\).

Also, by \(-x + e + x = -y + e + y\), we have that \(x + (-x) + e + x + (-y) = x + (-y) + e + y + (-y)\). Now,
\[
x + (-x) + e + x + (-y) = e + x + (-x) + x + (-y) = e + x + (-y)
\]
and
\[
x + (-y) + e + y + (-y) = x + (-y) + y + (-y) + e = x + (-y) + e,
\]
and so \(e + x + (-y) = x + (-y) + e\) for every \(e\) in \(E^+(S)\), that is, \(x + (-y) \in E^+\), also, \(-x + e + x + (-y) + y = -x + x + (-y) + e + y\). But \(-x + e + x + (-y) + y = -x + e + x + (-y) + x = -x + e + x\), and similarly \(-x + x + (-y) + e + y = -y + e + y\). Thus \(-x + e + x = -y + e + y\) for every \(e\) in \(E^+(S)\) and so \((x, y) \in \mu\) as required.

This completes the proof. \(\square\)

The next corollary is easy to verify.

Corollary 2.3. Let \(S\) be an inverse semiring, and let \(\mu\) be the maximum additive idempotent separating congruence on \(S\). Then \(\mu = \varepsilon_S\), the identical congruence on \(S\), if and only if \(E^+(S)\) is self-centralising in \(S\).

3 The minimum skew-ring congruence

For an arbitrary inverse semiring \(S\), the characterization of the minimum skew-ring congruence \(\sigma\) is given by the following lemma.

Lemma 3.1. If \(S\) is an inverse semiring, then the minimum skew-ring congruence \(\sigma\) on \(S\) is given by
\[
\sigma(S) = \{(x, y) \in S \times S | e + x = e + y \text{ for some } e \in E^+(S)\}
\]
or
\[
\sigma(S) = \{(x, y) \in S \times S | x + e = y + e \text{ for some } e \in E^+(S)\}.
\]

An alternative characterization of \(\sigma\) is provided by the next theorem.
Theorem 3.2. Let $S$ be an inverse semiring, and let $\sigma$ be the minimum skew-ring congruence on $S$. Then $(x, y) \in \sigma$ if and only if $x + (-y) \in E^+\omega$.

Proof. Suppose first that $e + x = e + y$ for some $e$ in $E^+(S)$. Then $e + x + (-y) = e + y + (-y) \in E^+(S)$. We have that $x + (-y) \leq e + x + (-y)$, and so $x + (-y) \in E^+\omega$.

Conversely, suppose that $x + (-y) \in E^+\omega$. Then there exists $f$ in $E^+(S)$ such that $x + (-y) \leq f$, i.e. such that $f + x + (-y) = f$. Now we denote $f + x + (-y) + y + (-x)$ by $e$, then $e \in E^+(S)$ and clearly $e + f = e$. Also,

$$
eq x + (-y) + y = e + f + y = e + y.$$ 

Thus this theorem is proved.

We require some preliminaries before investigating the nature of $\sigma \lor \mu$.

Lemma 3.3. Let $N$ be a closed normal ideal on an inverse semiring $S$. Then the relation $\rho_N$ defined by the rule that $(x, y) \in \rho_N$ if and only if $x + (-y) \in N$ is a semiring congruence on $S$.

Proof. Since $x + (-x) \in E^+(S) \subseteq N$, we have that $\rho_N$ is reflexive. It is symmetric since $y + (-x) = -(x + (-y)) \in N$. Suppose now that $x + (-y), y + (-z) \in N$. Then $x + (-y) + y + (-z) \in N$ since $N$ is a subsemiring. But $x + (-z) \leq x + (-y) + y + (-z)$ since $-y + y$ is an additive idempotent, and so $x + (-z) \in N$ since $N$ is closed. Thus $\rho_N$ is transitive.

Now suppose that $x + (-y) \in N$ and that $z$ is an arbitrary element of $S$. Then $(z + x) + (-(z + y)) = z + x + (-y) + (-z) \in N$ since $N$ is self-conjugate. Also,

$$(x + z) + (-(y + z)) = x + z + (-z) + (-y) = (x + (-y)) + (y + z + (-z) + (-y)) \in N + E^+(S)$$

so $x + z + (-y + z) \in N$. We also have $zx + (-zy) = z(x + (-y)) \in N$ and $xz + (-yz) = (x + (-y))z \in N$ since $N$ is an ideal of $S$. Thus $\rho_N$ is a semiring congruence.

Lemma 3.4. If $H$ is a normal ideal of an inverse semiring $S$, then so is $H\omega$.

Proof. Let $x, y \in H\omega$ and $s \in S$, and let $h$ and $k$ be the element of $H$ such that $x \leq h$ and $y \leq k$. Form the compatibility of the order relation it now follows that $x + y \leq h + k \in H$ and $xs \leq hs \in H, sx \leq sh \in H$ since $H$ is an ideal of $S$, so $x + y \in H\omega, xs \in H\omega, sx \in H\omega$, thus $H\omega$ is an ideal of $S$. 


We have that \(-x \leq -h \in H\), hence \(-x \in H\omega\), and \(E^+(S) \subseteq H \subseteq H\omega\). Finally, if \(z\) is an arbitrary element of \(S\), it follows from the compatibility of the order relation, that \(z + x + (-z) \leq z + h + (-z) \in H\); hence \(z + x + (-z) \in H\omega\). Now we complete the proof of the lemma. 

**Lemma 3.5.** Let \(S\) be an inverse semiring. Then the centraliser \(E^+\zeta\) of \(E^+(S)\) is a normal ideal of \(S\).

**Proof.** It is clear that \(x + y \in E^+\zeta\) if \(x\) and \(y\) do. If \(x \in E^+\zeta\) and \(s \in S\), then \(x + e = e + x\) for every \(e\) in \(E^+(S)\). We have that \(-(x + e) = -(e + x)\), i.e. \(e + (-x) = (-x) + e\); hence \(-x \in E^+\zeta\), we also have \(es + xs = xs + es, se + sx = sx + se\). Since \(es\) and \(se\) are both arbitrary additive idempotents of \(S\), then \(E^+\omega\) is an ideal of \(S\). Let \(z \in S\), then

\[
\begin{align*}
z + x + (-z) + e &= z + x + (-z) + z + e = z + x + (-z) + e + z + (-z) \\
&= z + (x + (-z) + e + z) + (-z) = z + (-z) + e + z + x + (-z) \\
&= e + z + x + (-z)
\end{align*}
\]

for every \(e \in E^+(S)\); hence \(z + x + (-z) \in E^+\zeta\). So \(E^+\zeta\) is a normal ideal of \(S\).

As an immediate consequence of the last two lemmas, we have

**Lemma 3.6.** If \(S\) is an inverse semiring, then \((E^+\zeta)\omega\) is a close normal ideal of \(S\).

Now it follows from Lemma 3.3 and 3.6 that the relation \(\rho_{(E^+\zeta)\omega}\) which from now on we shall denote simply by \(\rho\) is a semiring congruence on \(S\).

The next theorem characterizes \(\sigma \vee \mu\).

**Theorem 3.7.** Let \(\sigma\) be the minimum skew-ring congruence and \(\mu\) the maximum additive idempotent separating congruence on an inverse semiring \(S\). Then the relation \(\rho\), defined by the rule that \((x, y) \in \rho\) if and only if \(x + (-y) \in (E^+\zeta)\omega\) is equal to \(\sigma \vee \mu\).

**Proof.** We have already remarked that \(\rho\) is a congruence on \(S\). Moreover, it follows immediately from Theorems 2.2 and 3.2 that \(\sigma \subseteq \rho\) and \(\mu \subseteq \rho\); hence \(\sigma \vee \mu \subseteq \rho\). It remains to prove that \(\rho \subseteq \sigma \vee \mu\). We prove in fact that \(\rho \subseteq \sigma \circ \mu \circ \sigma\), which is clearly sufficient.

Suppose that \((x, y) \in \rho\). Then there exists \(z \in E^+\zeta\) such that \(x + (-y) \leq z\).

Let \(u = z + y\) and \(v = -z + z + y\). Then \(x + (-u) = x + (-y + (-z)) \leq z + (-z) \in E^+(S)\) and so \(x + (-u) \in E^+\omega\). Thus \((x, u) \in \sigma\). Also,

\[
-v + v = -y + (-z) + z + (-z) + z + y = -y + (-z) + z + y = -u + u
\]
and for every $e$ in $E^+(S)$,

$$u + (-v) + e = z + y + (-y) + (-z) + z + e = z + e + y + (-y) + (-z) + z = e + z + y + (-y) + (-z) + z = e + u + (-v).$$

Thus $u + (-v) \in E^+\zeta$ and so, by Theorem 2.2, we have that $(u, v) \in \mu$. Finally, $v + (-y) = -z + (z + y + (-y)) \in E^+(S) \subseteq E^+\omega$, and so $(v, y) \in \sigma$. Now we have that

$$(x, u) \in \sigma, \quad (u, v) \in \mu, \quad (v, y) \in \sigma,$$

and so $(x, y) \in \sigma \circ \mu \circ \sigma$ as required. This completes the proof of the theorem. \qed

On the end of this paper, we give a characterization of the congruence $\xi$.

**Theorem 3.8.** Let $S$ be an inverse semiring, $\eta$ a distributive lattice congruence on $S$. Then the relation $\xi$ defined as follow:

$$a \xi b \iff a \eta b \text{ and } e + a = e + b \text{ for some } e + e = e \eta a$$

is a congruence such that $S/\xi$ is a distributive lattice of skew-rings.

**Proof.** It is easily seen that $\xi$ is an equivalence relation on $S$. Let $a \xi b$ and $x$ be in $S$. Then $a \eta b$ and $e + a = e + b$ for some $e + e = e \eta a$. Since $\eta$ is a congruence, $a + x \eta b + x$. Let $f$ be any additive idempotent such that $f \eta x$. Then $e + f \eta a + x$ and

$$(e+f)+(a+x) = f+(e+a)+x = f+(e+b)+x = (f+e)+(b+x) = (e+f)+(b+x);$$

therefore, $a + x \xi b + x$. On the other hand, $x + a \eta x + b$ and $x + e \eta x + a$. Thus, since $\eta$ is a distributive lattice congruence, then, $x + e + (-x) \eta x + a$. In addition,

$$(x+e+(-x))+(x+a) = x+e+(-x)+a = x+e+a = x+e+b = (x+e+(-x))+(x+b);$$

that is, $x + a \xi x + b$. We also have $a \eta bx$ and $ex + ax = ex + bx$ for some $ex + ex = ex \eta ax$, $x \eta bx$ and $xe + xa = xe + xb$ for some $xe + xe = xe \eta xa$. Therefore, $ax \xi bx$ and $xa \xi xb$, $\xi$ is a congruence on $S$.

To see that $S/\xi$ is a distributive lattice of skew-rings, it is sufficient to show that $a + (-a) \xi (-a) + a$ for all $a$ in $S$. But this is clear by letting $e = (a+(-a))+(-a+a)$. Now we have completed the proof of this theorem. \qed
References


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