The Categories of $L$-FTOP and $L$-FINT

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Abstract

In this paper, the category of $L$-FINT is topological category over Set. We investigate the functorial relations between $L$-FTOP and $L$-FINT where $L$ is a strictly two-sided, commutative quantale lattice.

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1 Introduction

Höhle and Šostak [5] introduced the notions of $L$-fuzzy topologies and $L$-fuzzy interior operators where $L$ is a cqm-lattice substituted for complete lattices or the two-point lattice $2 = \{0, 1\}$ in the definitions of $L$-(fuzzy) topologies and $L$-(fuzzy) closure spaces in [2-4, 6, 9].

In this paper, we show the existence of initial $L$-fuzzy interior spaces. From this fact, the category of $L$-FINT (the category of $L$-fuzzy interior spaces and $LF$-interior maps) is topological category over Set. We investigate the functorial relations between $L$-FTOP (the category of $L$-fuzzy topological spaces and $LF$-continuous maps) and $L$-FINT where $L$ is a strictly two-sided, commutative quantale lattice. The functor from $L$-FTOP to $L$-FINT is left adjoint. Furthermore, we can define subspaces and products of $L$-fuzzy interior spaces. For general categorical background, we refer to Adámek et al.[1] and Rodabaugh [8].
2 Preliminaries

Definition 2.1 [5, 7, 10] A triple \((L, \leq, \odot)\) is called a strictly two-sided, commutative quantale (stsc-quantale, for short) iff it satisfies the following properties:

(L1) \(L = (L, \leq, 1, 0)\) is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) \((L, \odot)\) is a commutative semigroup;

(L3) \(a = a \odot \top\), for each \(a \in L\);

(L4) \(\odot\) is distributive over arbitrary joins, i.e.

\[
(\bigvee_{i \in \Gamma} a_i) \odot b = \bigvee_{i \in \Gamma} (a_i \odot b).
\]

Example 2.2 [5, 7, 10] (1) Each frame is a stsc-quantale. In particular, the unit interval \([0, 1], \leq, \lor, \land, 0, 1\) is a stsc-quantale.

(2) The unit interval with a left-continuous t-norm \(t\), \([0, 1], \leq, t\), is a stsc-quantale.

(3) Every GL-monoid is a stsc-quantale.

(4) Define a binary operation \(\odot\) on \([0, 1]\) by \(x \odot y = \max\{0, x + y - 1\}\). Then \([0, 1], \leq, \odot\) is a stsc-quantale.

Lemma 2.3 [5, 7] Let \((L, \leq, \odot)\) be a stsc-quantale. If \(f : X \to Y\), then we have: for \(\mu, \mu_i \in L^X\) and \(\nu, \nu_i \in L^Y\),

(1) \(\nu \geq f^-(f^-(\nu))\) with equality if \(f\) is surjective.

(2) \(\mu \leq f^-(f^-(\mu))\) with equality if \(f\) is injective.

(3) \(f^-(\nu_1 \odot \nu_2) = f^-(\nu_1) \odot f^-(\nu_2)\).

(4) \(f^-(\mu_1 \odot \mu_2) \leq f^-(\mu_1) \odot f^-(\mu_2)\) with equality if \(f\) is injective.

All algebraic operations on \(L\) can be extended pointwise to the set \(L^X\) as follows: for all \(x \in X\),

(1) \(f \leq g\) iff \(f(x) \leq g(x)\);

(2) \((f \circ g)(x) = f(x) \circ g(x)\);

(3) \(\overline{\alpha}(x) = \alpha\), for \(\alpha \in L\).

Definition 2.4 [5, 7] A function \(T : L^X \to L\) is called an \(L\)-fuzzy topology on \(X\) if it satisfies the following conditions:

(O1) \(T(\emptyset) = T(\emptyset) = 1\).

(O2) \(T(\lambda_1 \odot \lambda_2) \geq T(\lambda_1) \odot T(\lambda_2), \forall \lambda_1, \lambda_2 \in L^X\).

(O3) \(T(\bigvee_{i \in \Gamma} \lambda_i) \geq \bigwedge_{i \in \Gamma} T(\lambda_i), \forall \{\lambda_i\}_{i \in \Gamma} \subset L^X\).

The pair \((X, T)\) is called an \(L\)-fuzzy topological space. Let \(T_1\) and \(T_2\) be \(L\)-fuzzy topologies on \(X\). We say that \(T_1\) is finer than \(T_2\) \((T_2\) is coarser than \(T_1)\) if \(T_2(\lambda) \leq T_1(\lambda)\) for all \(\lambda \in L^X\).
Definition 2.5 [5,7] A map $\mathcal{I} : L^X \times L \to L^X$ is called an $L$-fuzzy interior operator on $X$ iff $\mathcal{I}$ satisfies the following conditions:

(I) $\mathcal{I}(\lambda, r) = \lambda, \forall r \in L$.

(II) $\mathcal{I}(\lambda, r) \leq \lambda, \forall r \in L$.

(III) If $\lambda \leq \mu$ and $r \leq s$, then $\mathcal{I}(\lambda, s) \leq \mathcal{I}(\mu, r)$.

(IV) $\mathcal{I}(\lambda \cup \mu, r \cup s) \geq \mathcal{I}(\lambda, r) \cup \mathcal{I}(\mu, s)$.

(V) $\mathcal{I}(\lambda, 0) = \lambda$.

The pair $(X, \mathcal{I})$ is called an $L$-fuzzy interior space.

An $L$-fuzzy interior space $(X, \mathcal{I})$ is called topological if

$$\mathcal{I}(\mathcal{I}(\lambda, r), r) \geq \mathcal{I}(\lambda, r), \forall \lambda \in L^X, r \in L.$$  

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be $L$-fuzzy interior operators on $X$. We say that $\mathcal{I}_1$ is finer than $\mathcal{I}_2$ ($\mathcal{I}_2$ is coarser than $\mathcal{I}_1$) if $\mathcal{I}_2(\lambda, r) \leq \mathcal{I}_1(\lambda, r)$ for all $\lambda \in L^X$ and $r \in L$.

Theorem 2.6 [5,7] Let $(X, \mathcal{I})$ be an $L$-fuzzy interior space. Define a map $\mathcal{I}_X : L^X \to L$ by

$$\mathcal{I}_X(\lambda) = \bigwedge \{r \in L \mid \lambda \leq \mathcal{I}(\lambda, r)\}.$$  

Then $\mathcal{I}_X$ is an $L$-fuzzy topology on $X$ induced by $\mathcal{I}$.

Definition 2.7 [1,8] Let $A$ and $B$ be categories. A functor $V : A \to B$ is called topological if every $V$-structured source $(f_i : B \to V(A_i, S_i))_{i \in I}$ has a unique $V$-initial lift $\bar{f}_i : (A, S) \to A_i)_{i \in I}$ such that $V(A, S) = B$ and $V(\bar{f}_i) = f_i$. The structure $S$ is called an $V$-initial structure on $A$ with respect to $(B, f_i, (A_i, S_i), I)$.

Definition 2.8 [1,8] Let $A$ and $B$ be categories. Let $G : A \to B$ be a functor and $B$ a $B$-object.

(1) A $G$-structured arrow $(f, A)$ with domain $B$ is called $G$-universal for $B$ if for each $G$-structured arrow $(f', A')$ with domain $B$ there exists a unique $A$-morphism $\bar{f} : A \to A'$ with $f' = G(\bar{f}) \circ f$.

(2) A functor $G$ is called left adjoint if for every $B$-object $B$ there exists a $G$-universal arrow with domain $B$.

Theorem 2.9 [5] The forgetful functor $V : L\text{-FTOP} \to \text{Set}$ defined by $V(X, \mathcal{T}) = X$ and $V(f) = f$ is topological.

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Theorem 3.1 Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space. Define a map $\mathcal{I}_T : L^X \times L \to L$ by

$$\mathcal{I}_T(\lambda, r) = \bigwedge \{\mu \in L^X \mid \mu \leq \lambda, \mathcal{T}(\mu) \geq r\}.$$  

Then (1) $\mathcal{I}_T$ is an $L$-fuzzy interior operator on $X$ with $\mathcal{I}_{T_T} = \mathcal{T}$.

(2) If $(X, \mathcal{I})$ is an $L$-fuzzy interior space, then $\mathcal{I}_{T_T} \leq \mathcal{I}$.
Proof. (1) $\mathcal{I}_T$ is an $L$-fuzzy interior operator in [5]. Suppose there exists $\lambda \in L^X$ such that $\mathcal{I}_T(\lambda) \not\leq \mathcal{T}(\lambda)$. Then there exists $r \in L$ with $\lambda \leq \mathcal{I}_T(\lambda, r)$ such that $\mathcal{T}(\lambda) \gtrless r$. Since $\lambda = \mathcal{I}_T(\lambda, r)$, by the definition of $\mathcal{I}_T(\lambda, r)$, we have $\mathcal{T}(\lambda) \gtrless r$. It is a contradiction. Hence $\mathcal{I}_T \leq \mathcal{T}$. Let $\mathcal{T}(\rho) \geq r$ be given. Then $\mathcal{I}_T(\rho) \geq r$. Hence $\mathcal{I}_T \geq \mathcal{T}$.

(2) Suppose there exists $\lambda \in L^X$ and $r \in L$ such that $\mathcal{I}_T(\lambda, r) \not\leq \mathcal{T}(\lambda, r)$. Then there exists $\rho \in L^X$ with $\rho \leq \lambda$ and $\mathcal{I}_T(\rho) > r$ such that $\rho \not\leq \mathcal{T}(\lambda, r)$. Since $\mathcal{I}_T(\rho) > r$, we have $\rho = \mathcal{T}(\rho, r)$. Hence $\mathcal{T}(\rho, r) \not\leq \mathcal{T}(\lambda, r)$. But $\rho \leq \lambda$ implies $\mathcal{I}(\rho, r) \leq \mathcal{T}(\lambda, r)$. It is a contradiction. Hence $\mathcal{I}_T \leq \mathcal{T}$.

Example 3.2 Let $X = \{a, b\}$ be a set and $L = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. Define operations $\circ$ on $L$ by $x \circ y = \max\{0, x + y - 1\}$. Then $(L, \leq, \circ)$ is a stsc quantale (ref.[5,7,10]). Let $\mu \in L^X$ as $\mu(a) = \frac{3}{4}$, $\mu(b) = \frac{1}{2}$. Define an $L$-fuzzy topology $\mathcal{T}: L^X \to L$ as follows:

$$\mathcal{T}(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in \{0, 1\}, \\
\frac{3}{4}, & \text{if } \lambda = \mu, \\
\frac{1}{2}, & \text{if } \lambda = \mu \circ \mu, \\
\frac{1}{2}, & \text{if } \lambda = \mu \circ \mu \circ \mu, \\
0, & \text{otherwise.}
\end{cases}$$

We obtain an $L$-fuzzy interior operator $\mathcal{I}_T: L^X \times L \to L$ as follows:

$$\mathcal{I}_T(\lambda, r) = \begin{cases} 
\overline{1}, & \text{if } \lambda = \overline{1}, r \in L, \\
\mu, & \text{if } \overline{1} \neq \lambda \geq \mu, r \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}, \\
\mu \circ \mu, & \text{if } \mu \not\leq \lambda \geq \mu \circ \mu, r \in \{\frac{1}{4}, \frac{1}{2}\} \\
\mu \circ \mu \circ \mu, & \text{if } \mu \circ \mu \not\leq \lambda \geq \mu \circ \mu \circ \mu, r \in \{\frac{1}{4}, \frac{1}{2}\}, \\
\lambda, & \text{if } r = 0, \\
\overline{0}, & \text{otherwise.}
\end{cases}$$

We obtain an $L$-fuzzy topology $\mathcal{T}_{\mathcal{I}_T}: L^X \to L$ with $\mathcal{T}_{\mathcal{I}_T} = \mathcal{T}$.

Definition 3.3 Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be $L$-fuzzy topological spaces. A map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called an $LF$-continuous map if $\mathcal{T}_2(\mu) \leq \mathcal{T}_1(f^{-}(\mu))$, for each $\mu \in L^Y$.

Definition 3.4 Let $(X, \mathcal{I}_1)$ and $(Y, \mathcal{I}_2)$ be $L$-fuzzy interior spaces. A map $f : (X, \mathcal{I}_1) \to (Y, \mathcal{I}_2)$ is called an $LF$-interior map if $f^{-}(\mathcal{I}_2(\lambda, r)) \leq \mathcal{I}_1(f^{-}(\lambda), r)$, for each $r \in L, \lambda \in L^Y$.

Theorem 3.5 Let $(X, \mathcal{I}_1)$ and $(Y, \mathcal{I}_2)$ be $L$-fuzzy interior spaces. Let $f : (X, \mathcal{I}_1) \to (Y, \mathcal{I}_2)$ be an $LF$-interior map. Then $f : (X, \mathcal{I}_{\mathcal{I}_1}) \to (Y, \mathcal{I}_{\mathcal{I}_2})$ is an $LF$-continuous map.
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**Proof.** Since $f^-(\mathcal{I}_2(\mu,r)) \leq \mathcal{I}_1(f^-(\mu),r)$, we have

\[
\mathcal{I}_{\mathcal{I}_2}(\mu) = \bigvee \{ r \in L \mid \mu \leq \mathcal{I}_2(\mu,r) \} \\
\leq \bigvee \{ r \in L \mid f^-(\mu) \leq \mathcal{I}_1(f^-(\mu),r) \} \\
= \mathcal{I}_{\mathcal{I}_1}(f^-(\mu)).
\]

**Theorem 3.6** Let $(X,\mathcal{I}_1)$ and $(Y,\mathcal{I}_2)$ be $L$-fuzzy topological spaces. A map $f : (X,\mathcal{I}_1) \to (Y,\mathcal{I}_2)$ is $L$-$\text{continuous}$ iff $f : (X,\mathcal{I}_{\mathcal{I}_1}) \to (Y,\mathcal{I}_{\mathcal{I}_2})$ is an $L$-$\text{interior}$ map.

**Proof.** By Theorems 3.1(2) and 3.6, we only show:

\[
f^-(\mathcal{I}_{\mathcal{I}_2}(\mu,r)) = f^-(\bigvee \{ \rho \in L^Y \mid \rho \leq \mu, \mathcal{I}_2(\rho) > r \}) \\
\leq f^-(\bigvee \{ f^-(\rho) \mid f^-(\rho) \leq f^-(\mu), \mathcal{I}_1(f^-(\rho)) > r \}) \\
= \mathcal{I}_{\mathcal{I}_1}(f^-(\mu),r).
\]

**Theorem 3.7** Let $\{(X_i,\mathcal{I}_i) \mid i \in \Gamma\}$ be a family of $L$-fuzzy topological spaces and $f_i : X \to X_i$ a map. Define a map $\mathcal{I} : L^X \times L \to L^X$ by

\[
\mathcal{I}(\lambda, r) = \bigvee \{ \odot_{k=1}^n f_{i_k}^-(\mathcal{I}_{i_k}(\lambda_{i_k}, r_k)) \mid \odot_{k=1}^n f_{i_k}^-(\lambda_{i_k}) \leq \lambda, \odot_{k=1}^n r_k \geq r \}
\]

for all finite subsets $K = \{i_1, ..., i_n\}$ of $\Gamma$. Then:

1. $\mathcal{I}$ is the coarsest $L$-fuzzy interior operator on $X$ for which all $f_i, i \in \Gamma$, are $L$-$\text{fuzzy}$-$\text{interior}$ maps.
2. If $\{(X_i,\mathcal{I}_i) \mid i \in \Gamma\}$ is a family of $L$-fuzzy topological interior spaces, then $\mathcal{I}$ is an $L$-fuzzy topological interior operator on $X$.

**Proof.** (1) First, we show that $\mathcal{I}$ is an $L$-fuzzy interior operator on $X$.

(I1) Since $f_i^-(\mathcal{I}) = \mathcal{I}$, we have $\mathcal{I}(\mathcal{I}, r) \geq f_i^-(\mathcal{I}(\mathcal{I}, r)) = \mathcal{I}$. Thus, $\mathcal{I}(\mathcal{I}, r) = \mathcal{I}$.

(I2) For all finite subsets $K = \{i_1, ..., i_n\}$ of $\Gamma$, we have

\[
\mathcal{I}(\lambda, r) = \bigvee \{ \odot_{k=1}^n f_{i_k}^-(\mathcal{I}_{i_k}(\lambda_{i_k}, r_k)) \mid \odot_{k=1}^n f_{i_k}^-(\lambda_{i_k}) \leq \lambda, \odot_{k=1}^n r_k \geq r \} \\
\leq \bigvee \{ \odot_{k=1}^n f_{i_k}^-(\mathcal{I}_{i_k}(\lambda_{i_k})) \mid \odot_{k=1}^n f_{i_k}^-(\lambda_{i_k}) \leq \lambda, \odot_{k=1}^n r_k \geq r \} \\
\leq \lambda.
\]

(I3) It is easily proved from the definition of $\mathcal{I}$.

(I4) Suppose there exist $\lambda, \mu \in L^X$ and $r, s \in L$ such that

\[
\mathcal{I}(\lambda, r) \odot \mathcal{I}(\mu, s) \not\leq \mathcal{I}(\lambda \odot \mu, r \odot s).
\]

By Definition 2.1(L4) and the definition of $\mathcal{I}(\lambda, r)$, there exists a finite subset $K = \{k_1, ..., k_p\}$ of $\Gamma$ with

\[
\odot_{i=1}^p f_{k_i}^-(\lambda_{k_i}) \leq \lambda, \odot_{i=1}^p r_{k_i} \geq r.
\]
such that
\[ \circ_{i=1}^{p} f_{k_i}^{-1}(\mathcal{I}_{k_i}(\lambda_{k_i}, r_i)) \circ \mathcal{I}(\mu, s) \not\subseteq \mathcal{I}(\lambda \circ \mu, r \circ s). \]

Again, the definition of \( \mathcal{I}(\mu, s) \), there exists a finite subset \( J = \{j_1, \ldots, j_q\} \) of \( \Gamma \) such that
\[ \circ_{m=1}^{q} f_{j_m}^{-1}(\mu_{j_m}) \leq \mu, \quad \circ_{m=1}^{q} s_m \geq s \]
such that
\[ \circ_{i=1}^{p} f_{k_i}^{-1}(\mathcal{I}_{k_i}(\lambda_{k_i}, r_i)) \circ \left( \circ_{m=1}^{q} f_{j_m}^{-1}(\mathcal{I}_{j_m}(\mu_{j_m}, s_m)) \right) \not\subseteq \mathcal{I}(\lambda \circ \mu, r \circ s). \quad (A) \]

On the other hand, for two finite subsets \( K = \{k_1, \ldots, k_p\} \) and \( J = \{j_1, \ldots, j_q\} \) of \( \Gamma \) such that
\[ \circ_{i=1}^{p} f_{k_i}^{-1}(\lambda_{k_i}) \leq \lambda, \quad \circ_{i=1}^{p} r_i \geq r, \quad \circ_{m=1}^{q} f_{j_m}^{-1}(\mu_{j_m}) \leq \mu, \quad \circ_{m=1}^{q} s_m \geq s, \]
we have
\[ \left( \circ_{i=1}^{p} f_{k_i}^{-1}(\lambda_{k_i}) \right) \circ \left( \circ_{m=1}^{q} f_{j_m}^{-1}(\mu_{j_m}) \right) \leq (\lambda \circ \mu), \]
\[ \left( \circ_{i=1}^{p} r_i \right) \circ \left( \circ_{m=1}^{q} s_m \right) \geq r \circ s. \]

Put \( M = K \cup L = \{m_1, \ldots, m_l\} \) with
\[
\rho_{m_n} = \begin{cases} 
\lambda_{m_n} & \text{if } m_n \in K - (K \cap J) \\
\mu_{m_n} & \text{if } m_n \in J - (K \cap J) \\
\lambda_{m_n} \circ \mu_{m_n} & \text{if } m_n \in K \cap J,
\end{cases}
\]
\[
t_n = \begin{cases} 
\lambda_{m_n} & \text{if } m_n \in K \cap J \\
\mu_{m_n} & \text{if } m_n \in J \cap K \\
r_n \circ s_n & \text{if } m_n \in K \cap J.
\end{cases}
\]

Since \( f_{m_n}^{-1}(\lambda_{m_n} \circ \mu_{m_n}) = f_{m_n}^{-1}(\lambda_{m_n}) \circ f_{m_n}^{-1}(\mu_{m_n}) \) for each \( m_n \in K \cap J \), we have
\[ \circ_{n=1}^{l} f_{m_n}^{-1}(\rho_{m_n}) \leq (\lambda \circ \mu), \quad \circ_{n=1}^{l} t_n \geq r \circ s. \]

Hence,
\[
\mathcal{I}(\lambda \circ \mu, r \circ s) \geq \circ_{n=1}^{l} f_{m_n}^{-1}(\mathcal{I}_{m_n}(\rho_{m_n}, t_n)) \\
= \left( \circ_{m_n \in (K \cup J) - (K \cap J)} f_{m_n}^{-1}(\mathcal{I}_{m_n}(\rho_{m_n}, t_n)) \right) \\
\circ \left( \circ_{m_n \in (K \cap J)} f_{m_n}^{-1}(\mathcal{I}_{m_n}(\lambda_{m_n} \circ \mu_{m_n}, r_n \circ s_n)) \right) \\
\geq \left( \circ_{m_n \in (K \cup J) - (K \cap J)} f_{m_n}^{-1}(\mathcal{I}_{m_n}(\rho_{m_n}, t_n)) \right) \\
\circ \left( \circ_{m_n \in (K \cap J)} f_{m_n}^{-1}(\mathcal{I}_{m_n}(\lambda_{m_n}, r_n)) \right) \\
\circ \left( \circ_{m_n \in (K \cap J)} f_{m_n}^{-1}(\mathcal{I}_{m_n}(\mu_{m_n}, s_n)) \right).
\]

It implies
\[ \mathcal{I}(\lambda \circ \mu, r \circ s) \geq \circ_{i=1}^{p} f_{k_i}^{-1}(\mathcal{I}_{k_i}(\lambda_{k_i}, r_i)) \circ \left( \circ_{m=1}^{q} f_{j_m}^{-1}(\mathcal{I}_{j_m}(\mu_{j_m}, s_m)) \right). \]
It is a contradiction for (A). Thus, $\mathcal{I}(\lambda, r) \circ \mathcal{I}(\mu, s) \leq \mathcal{I}(\lambda \circ \mu, r \circ s)$.

Second, for each $\lambda_i \in L^{X_i}$, one family $\{f_i^{-}(\lambda_i)\}$ and $i \in \Gamma$, we have

$$\mathcal{I}(f_i^{-}(\lambda_i), r) \geq f_i^{-}(\mathcal{I}(\lambda_i, r)).$$

Thus, each $i \in \Gamma$, $f_i : (X, \mathcal{I}) \to (X_i, \mathcal{I}_i)$ is an LF-interior map.

Finally, let $f_i : (X, \mathcal{I}_0) \to (X_i, \mathcal{I}_i)$ be an LF-interior map for each $i \in \Gamma$.

Since for each $i \in \Gamma$ and $\lambda_i \in L^{X_i}$,

$$\mathcal{I}_0(f_i^{-}(\lambda_i), r) \geq f_i^{-}(\mathcal{I}_i(\lambda_i, r)),$$

for all finite subsets $K = \{i_1, ..., i_n\}$ of $\Gamma$, we have

$$\mathcal{I}(\lambda, r) = \bigvee \{\bigcirc_{k=1}^n f_{i_k}^{-}(\mathcal{I}_{i_k}(\lambda_{i_k}, r_k)) \mid \bigcirc_{k=1}^n f_{i_k}^{-}(\lambda_{i_k}) \leq \lambda, \ \bigcirc_{k=1}^n r_k \geq r \}$$

$$\leq \bigvee \{\bigcirc_{k=1}^n \mathcal{I}_0(f_{i_k}^{-}(\lambda_{i_k}), r_k) \mid \bigcirc_{k=1}^n f_{i_k}^{-}(\lambda_{i_k}) \leq \lambda, \ \bigcirc_{k=1}^n r_k \geq r \}$$

$$\leq \bigvee \{\mathcal{I}_0(\bigcirc_{k=1}^n f_{i_k}^{-}(\lambda_{i_k}), \bigcirc_{k=1}^n r_k) \mid \bigcirc_{k=1}^n f_{i_k}^{-}(\lambda_{i_k}) \leq \lambda, \ \bigcirc_{k=1}^n r_k \geq r \}$$

(by Definition 2.5(4))

$$\leq \mathcal{I}_0(\lambda, r).$$

Hence $\mathcal{I}$ is the coarsest $L$-fuzzy interior operator on $X$.

(2) Suppose there exist $\lambda \in L^X$ and $r \in L$ such that

$$\mathcal{I}(\mathcal{I}(\lambda, r), r) \nleq \mathcal{I}(\lambda, r).$$

By the definition of $\mathcal{I}(\lambda, r)$, there exists a family $\{f_{i_k}^{-}(\lambda_{i_k}) \mid \bigcirc_{k=1}^n f_{i_k}^{-}(\lambda_{i_k}) \leq \lambda, \ \bigcirc_{k=1}^n r_k \geq r \}$ such that

$$\mathcal{I}(\mathcal{I}(\lambda, r), r) \nleq \bigcirc_{k=1}^n f_{i_k}^{-}(\mathcal{I}_{i_k}(\lambda_{i_k}, r_k)).$$

On the other hand, since

$$\bigcirc_{k=1}^n f_{i_k}^{-}(\lambda_{i_k}) \leq \lambda, \ \bigcirc_{k=1}^n r_k \geq r,$$

we have

$$\mathcal{I}(\lambda, r) \geq \bigcirc_{k=1}^n f_{i_k}^{-}(\mathcal{I}_{i_k}(\lambda_{i_k}, r_k)) \ \bigcirc_{k=1}^n r_k \geq r.$$

Again, by the definition of $\mathcal{I}(\mathcal{I}(\lambda, r), r)$,

$$\mathcal{I}(\mathcal{I}(\lambda, r), r) \geq \bigcirc_{k=1}^n f_{i_k}^{-}(\mathcal{I}_{i_k}(\lambda_{i_k}, r_k), r_k) \ \bigcirc_{k=1}^n r_k \geq r.$$

because $\mathcal{I}_{i_k}$ is an $L$-fuzzy topological interior operator. It is a contradiction.

**Theorem 3.8** The forgetful functor $W : L\text{-}	ext{FINT} \to \text{Set}$ defined by $W(X, \mathcal{I}) = X$ and $W(f) = f$ is topological.
Proof. Every $W$-structured source $(f_i : X \to W(X_i, \mathcal{I}_i))_{i \in \Gamma}$ has a unique $W$-initial lift $(f_i : (X, \mathcal{I}) \to (X_i, \mathcal{I}_i))_{i \in \Gamma}$ where $\mathcal{I}$ in Theorem 3.7.

Using Theorems 3.7 and 3.8, we obtain the following definition.

**Definition 3.9** Let $\{(X_i, \mathcal{I}_i)\}_{i \in \Gamma}$ be a family of $L$-fuzzy interior spaces, $X$ a set and $f_i : X \to X_i$ a function, for each $i \in \Gamma$. The initial $L$-fuzzy interior operator on $X$ with respect to $(X, f_i, (X_i, \mathcal{I}_i), \Gamma)$ is the coarsest $L$-fuzzy interior operator on $X$ for which all $f_i$, $i \in \Gamma$, are $LF$-interior maps.

**Corollary 3.10** Let $(Y, \mathcal{I}_Y)$ be an $L$-fuzzy interior space and $f : X \to Y$ a map. Define a map $\mathcal{I} : L^X \times L \to L^X$ by

$$\mathcal{I}(\lambda, r) = \bigvee \{ \circ_{k=1}^n f^-(\mathcal{I}_Y(\lambda_{i_k}, r_{k})) \mid \circ_{k=1}^n f^-(\lambda_{i_k}) \leq \lambda, \circ_{k=1}^n r_{k} \geq r \}$$

for all finite subsets $K = \{i_1, ..., i_n\}$. Then $\mathcal{I}$ is the coarsest $L$-fuzzy interior operator on $X$ for which $f$ is an $LF$-interior map such that

$$\mathcal{I}(\lambda, r) = \bigvee \{ f^-(\mathcal{I}_Y(\mu, r)) \mid f^-(\mu) \leq \lambda \}.$$ 

**Proof.** From Theorem 3.7 and the definition of $\mathcal{I}(\lambda, r)$, we only show:

$$\mathcal{I}(\lambda, r) \leq \bigvee \{ f^-(\mathcal{I}_Y(\mu, r)) \mid f^-(\mu) \leq \lambda \}.$$ 

Suppose $\mathcal{I}(\lambda, r) \not\subseteq \bigvee \{ f^-(\mathcal{I}_Y(\mu, r)) \mid f^-(\mu) \leq \lambda \}$. Then there exists a finite subsets $K = \{i_1, ..., i_n\}$ with $\circ_{k=1}^n f^-(\lambda_{i_k}) \leq \lambda, \circ_{k=1}^n r_{k} \geq r$ such that

$$\circ_{k=1}^n f^-(\mathcal{I}_Y(\lambda_{i_k}, r_{k})) \not\subseteq \bigvee \{ f^-(\mathcal{I}_Y(\mu, r)) \mid f^-(\mu) \leq \lambda \}.$$ 

On the other hand, since $\circ_{k=1}^n f^-(\lambda_{i_k}) = f^-(\circ_{k=1}^n \lambda_{i_k}) \leq \lambda$, we have

$$\bigvee \{ f^-(\mathcal{I}_Y(\mu, r)) \mid f^-(\mu) \leq \lambda \} \geq f^-(\circ_{k=1}^n \mathcal{I}_Y(\lambda_{i_k}, r_{k})) = \circ_{k=1}^n f^-(\mathcal{I}_Y(\lambda_{i_k}, r_{k})).$$

It is a contradiction. Hence the result follows.

**Definition 3.11** Let $(X, \mathcal{I}_X)$ be an $L$-fuzzy interior space and $A$ a subset of $X$. The pair $(A, \mathcal{I})$ is said to be a subspace of $(X, \mathcal{I}_X)$ if it is endowed with the initial $L$-fuzzy interior operator with respect to $(A, i, (X, \mathcal{I}_X))$ where $i$ is the inclusion function. From Corollary 3.10, we define the function $\mathcal{I} : L^A \times L \to L^A$ on $A$ by

$$\mathcal{I}(\lambda, r) = \bigvee \{ i^-(\mathcal{I}_X(\mu, r)) \mid i^-(\mu) \leq \lambda \}.$$
Definition 3.12 Let $X = \prod_{i \in \Gamma} X_i$ be the product of the sets from family \{(X_i, \mathcal{I}_i) \mid i \in \Gamma\} of $L$-fuzzy interior spaces. The initial $L$-fuzzy interior operator $\mathcal{I} = \otimes \mathcal{I}_i$ on $X$ with respect to the family \{\pi_i : X \to (X_i, \mathcal{I}_i) \mid i \in \Gamma\} of all projection maps is called the product $L$-fuzzy interior operator of \{\mathcal{I}_i \mid i \in \Gamma\}, and $(X, \otimes \mathcal{I})$ is called the product $L$-fuzzy interior space.

Example 3.13 Let $X = \{a, b\}$ be a set. Define an operations $\odot$ in Example 2.2(4). Let $\mu, \rho \in [0, 1]^X$ as follows:

$$\mu(a) = 0.6, \mu(b) = 0.6, \rho(a) = 0.1, \rho(b) = 0.5.$$  

Define $[0, 1]$-fuzzy interior operators $\mathcal{I}_i : [0, 1]^X \times [0, 1] \to [0, 1]$ as follows:

$$\mathcal{I}_1(\lambda, r) = \begin{cases} \top, & \text{if } \lambda = \top, \ r \in [0,1], \\ \mu, & \text{if } \top \neq \lambda \geq \mu, \ 0 < r \leq 0.6, \\ \mu \odot \mu, & \text{if } \mu \leq \lambda \geq \mu \odot \mu, \ 0 < r \leq 0.3, \\ \lambda, & \text{if } r = 0, \\ \emptyset, & \text{otherwise}. \end{cases}$$

$$\mathcal{I}_2(\lambda, r) = \begin{cases} \top, & \text{if } \lambda = \top, \ r \in [0,1], \\ \rho, & \text{if } \top \neq \lambda \geq \rho, \ 0 < r \leq 0.5, \\ \lambda, & \text{if } r = 0, \\ \emptyset, & \text{otherwise}. \end{cases}$$

From Theorem 3.7, we obtain the initial $[0, 1]$-fuzzy interior operator $\mathcal{I}$ on $X$ finer than $\mathcal{I}_1$ and $\mathcal{I}_2$ as follows:

$$\mathcal{I}(\lambda, r) = \begin{cases} \top, & \text{if } \lambda = \top, \ r \in [0,1], \\ \mu, & \text{if } \top \neq \lambda \geq \mu, \ 0 < r \leq 0.6, \\ (\mu \odot \mu) \lor \rho, & \text{if } \mu \leq \lambda \geq (\mu \odot \mu) \lor \rho, \ 0 < r \leq 0.3, \\ \rho, & \text{if } (\mu \odot \mu) \lor \rho \leq \lambda \geq \rho, \ 0 < r \leq 0.3, \\ \top \neq \lambda \geq \rho, \ 0.3 < r \leq 0.5, \\ \mu \odot \mu, & \text{if } (\mu \odot \mu) \lor \rho \leq \lambda \geq \mu \odot \mu, \ 0 < r \leq 0.3, \\ \mu \odot \rho, & \text{if } \mu \odot \mu \leq \lambda \geq \mu \odot \rho, \ 0 < r \leq 0.1, \\ \lambda, & \text{if } r = 0, \\ \emptyset, & \text{otherwise}. \end{cases}$$

Theorem 3.14 Define maps as follows:

$D : L\text{-FTOP} \to L\text{-FINT}$ by $D(X, T) = (X, \mathcal{I}_T)$ and $D(f) = f$,

$E : L\text{-FINT} \to L\text{-FTOP}$ by $E(X, \delta) = (X, \mathcal{T}_\delta)$ and $E(g) = g$. Then:

(1) $E$ is a functor. Moreover, $D$ is full, faithful and embedding functor.

(2) The functor $D : L\text{-FTOP} \to L\text{-FINT}$ is a left adjoint of $E$.

Proof. (1) From Theorem 3.5, $E$ is a functor. From Theorems 3.1 and 3.6, since $E \circ D(X, T) = (X, \mathcal{I}_T) = (X, T)$, it easily proved.
(2) Let \((X, \mathcal{I})\) be in \(L\)-\textbf{FINT}. Since \(D \circ E(X, \mathcal{I}) = (X, \mathcal{I}_{T_I})\) and \(\mathcal{I} \geq \mathcal{I}_{T_I}\) from Theorem 3.1(2), then an identity map \(id_X : (X, \mathcal{I}) \to D \circ E(X, \mathcal{I}) = (X, \mathcal{I}_{T_I})\) is an \(LF\)-interior map. In fact, \((id_X, (X, \mathcal{I}_{T_I}))\) is a \(D\)-universal arrow for \((X, \mathcal{I})\). Let \(f : (X, \mathcal{I}) \to D(Y, \mathcal{T})\) be an \(LF\)-interior map in \(L\)-\textbf{FINT}. Since \(E\) is a functor, \(f = E(f) : (X, \mathcal{T}_{T_I}) \to E \circ D(Y, \mathcal{T}) = (Y, \mathcal{T})\) is \(LF\)-continuous. Thus \(f = D(f) \circ id_X\). Hence the result follows.

**References**


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