Icosahedral Symmetry in the MSSM

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Abstract

In this note we discuss the relationship between the 15 synthemes in the icosahedron and the 40 particles in the Minimal Supersymmetric Standard Model, or MSSM, including the vector $W^{\pm}$, $Z$ bosons, 8 gluons and the photon. Because the icosahedron is 6-symmetric there can be jets in 4-, 5-, and 6-space in a constant time frame, but only those projected onto a 3- space are actually observed. In particular the 6 dimensional Planck space is isomorphic to the Lie algebra $E_6$, and yields a cubic surface with 27 vertices labeling the Standard Model. A tetrahedron labeled by the up and down quarks is shown in Fig.3. This is part of an orbifold that may be covered by a World-brane shown in Fig.4 when the quarks unite to form protons resulting in Inflation of the order of $10^{20}$ just after the Big Bang with accompanying cooling. This union could also account for Dark Matter still created.

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1 Introduction

A duad is a rectangle connecting 4 vertices of the regular icosahedron where the golden ratio is length of the long side divided by the length of the short side. The short sides are opposite edges and since there are 30 edges there are 30 duads. Then a syntheme is a set of 3 duads, no two having an axis in common, so there are also 15 synthemes in the Dirac algebra which this note shows to be labeled by the 15 sets of commuting elements that correspond to particles in the MSSM. In Fig.1 there are examples of 5 duads that are mutually perpendicular and a further 10 duads that share a common line of intersection (cf Baez [1])
The icosahedron is isomorphic to the Lie algebra $E_8$ which contains the Witting polytope with just 40 tetrahedra and in Section 3 these will be mapped into the 15 synthemes. Because the Witting polytope and icosahedron are 6-symmetric [1,5] there will be particles and possibly jets in a 6-space.

The connection with particle physics derives from the fundamental nuclear representation equation (1) of the Lorentz group [6] which Barth and Nieto [2] related to 3 pairs of 'fix-lines' in the $\pm i$ eigenspaces defined by $E^2_{\mu\nu} = -1$. The 6 fix-lines are the edges of a fundamental tetrahedron so (1) implies tetrahedral symmetry governed by the quaternion group $T_d$ (cf Coxeter [5]). Furthermore $T_d$ is isomorphic to the exceptional Lie algebra $E_6$ by the MacKay correspondence [14] and Fig.2 shows the famous 27 lines on a cubic surface that correspond to the 27 fundamental weights of $E_6$ (cf Hunt [12], Section 4.1.1). These weights were associated with members of the Standard Model by Slansky [15] and a possible allocation of the 18 quarks $u, d, s$ associated with the proton, but these do not enjoy tetrahedral symmetry so there are only 6 tetrahedra in $E_6$. The 9 vertices of Fig.3 carry Coxeter’s labels $(\lambda, \mu, \nu)$ for the powers of $\omega = \exp(2\pi i/3)$ where $\lambda, \mu, \nu$ can independently assume the values 0,1,2 [5]. Thus the apex of each equilateral triangle is just the previous one multiplied by $\omega$ and is a SO(6) rotation through 120 degrees. Referring to Fig.3 this twist can rotate separately $u(012) \rightarrow u(023) \rightarrow \overline{d}(031); \overline{d}(013) \rightarrow \overline{\pi}(021) \rightarrow \overline{\pi}(032); e(011) \rightarrow e^+(022) \rightarrow \nu_e(033)$ equivalent to $su_3 \times su_3 \times su_3$. But $su_3 \times su_3 \times su_3$ is a subalgebra of $E_6$ by [13] so $E_6/(su_3 \times su_3 \times su_3)$ has $su_3$ holonomy on a 6-d homogenous cubic surface with $su_3 \times su_3 \times su_3$ gauge. A hyperplane section of the homogenous Segre cubic in 6-d is the Clebsch diagonal cubic discussed in [9] which generates the Riemann surface of Fig.4.

Returning to the regular icosahedron this is isomorphic to $E_8$ by the MacKay correspondence [14] and consists of a lattice of 8 shells, the innermost being the Witting polytope that is 6-symmetric [5] with 40 tetrahedra that may be allocated as follows.
Equation (1) describes a rotation \( E_{23} \) about \( x_1 \) so there are another two possible rotations about \( x_2, x_3 \). The accompanying synthemes are also in the first column of Table 1. This accounts for 18 tetrahedra but because \( E_8 \) is \( \mathbb{Z}_2 \)-graded there could be another 18 supersymmetric partners in an anti-de-Sitter 5-space constituting the second column. The third column in 6-space is thought to describe the \( W^\pm, Z \) bosons. There remain 3 tetrahedra allocated to 8 gluons and the photon.

In the next Section we will derive the relationship between a nucleon and \( E_6 \) and then further discuss the MSSM and gluon representations as well as the top, bottom and charm quarks. In particular derive the ratio of quark and gluon jets.

Finally Inflation will be discussed in Section 4.

\section{The Fundamental Idempotent Relation}

The 2-form
\[
\frac{1}{4} \Psi = (iE_4 \psi_1 + E_{23} \psi_2 + E_{14} \psi_3 + E_{05} \psi_4)e
\]  
(1)
is a minimal left ideal of the Dirac ring describing spin about \( x_3 \) and is an irreducible representation of the Lorentz group [6]. It has been used to model a nucleon because \( E_{23}, E_{05} \) are rotational operators of spin and isospin through angles \( \psi_2, \psi_4 \) while \( E_{14} \) is a parity operator completing triality. \( iE_4 \psi_1 \) is the identity operator for rotations mod 2\( \pi \).

Eddington’s E-numbers are mapped into the \( 4 \times 4 \) Dirac matrices by
\[
\begin{align*}
\gamma_\nu &= iE_0 \nu, \\
E_{\mu\nu} &= E_{\rho\mu}E_{\rho\nu} = -E_{\nu\mu}, \\
E_{\mu\nu}^2 &= -1, \\
E_{\mu\nu}E_{\sigma\tau} &= E_{\sigma\tau}E_{\mu\nu} = iE_{\lambda\rho}, \mu < \nu = 1, \ldots, 5
\end{align*}
\]  
(2)
where \( E_4 \) is the unit matrix. Two more equivalent representations may be obtained by a cyclic interchange of the indices 1,2,3 to express rotations about \( x_2, x_3 \).

It is proved in [9] that
\[
(\Psi/4)^2 = (\Psi/4)iE_4 \psi_1
\]  
(3)
which is idempotent if \( iE_4 \psi_1 = 1 \).

Following Barth and Nieto we associate the u,u;u,d lines of the principal triangle of Fig.3 with the operators \( E_{05} = -E_{50} \) with isospin \( \pm 1/2 \) so that this recipe yields the quark charges 1/6 \( \pm 1/2 \). The third edge is assigned to the parity operator \( E_{14} \) so that a change \( E_{14} \rightarrow E_{41} \) from a Left to a Right handed quark is accompanied by charge conjugation. The 3 remaining skew-lines meeting at 0 are labeled by \( E_{41} \) and \( E_{23}, E_{32} \) with spin \( \pm 1/2 \).
In particular the asyzygy that inverts vectors meeting at O will cause $E_{05} \leftrightarrow E_{50}$ thus changing quarks into anti-quarks.

Equation (3) is idempotent up to $i E_4 \psi_1$, but long ago Boole [3] showed that idempotency is a fundamental theorem of logic so (1) satisfies our basic requirement of logical Lorentz invariance. Furthermore tetrahedral symmetry gives us Figs.2,3 and the Standard Model.

3 The 15 Synthemes in $E_8$

Apart from $E_{05}$ in equation (1) there are two more commuting operators $E_{04}$ and $E_{45}$ that define rotations in iso- or charge space. So the columns of Table 1 all describe rotations $E_{12}, E_{31}, E_{23}$ in 3-space but in 3 different charge spaces each also characterized by 3 vectors $(E_{14}, E_{24}, E_{34}); (E_{15}, E_{25}, E_{35}); (E_{01}, E_{02}, E_{03})$ in 4-,5- and 6-space.

Table 1

<table>
<thead>
<tr>
<th>12</th>
<th>34</th>
<th>05</th>
<th>12</th>
<th>35</th>
<th>04</th>
<th>12</th>
<th>03</th>
<th>45</th>
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<tbody>
<tr>
<td>31</td>
<td>24</td>
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<td>25</td>
<td>04</td>
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<td>04</td>
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</tbody>
</table>

The first columns of the Table account for 36 tetrahedra, namely 18 associated with the Standard Model in 3 dimensions, and a further 18 for their supersymmetric partners. Although the top, bottom and Charm quarks are not shown in Fig 2, they are believed to be simply the low energy quarks excited to higher energies and entropy as described by de Wet [8] where this idea checks with experimental mass ratios when particle representations are assigned to subalgebras of $E_8$ by Manivel [13]. In this way the very high masses of the top and bottom quarks can be attributed to entropy increase which also accounts for the greater mass of the strange particles.

The third column of Table 1 is associated with the $W^\pm, Z^0$ vector bosons at the apices of a tritangent on a tetrahedron that is not part of the orbifold illustrated in Fig.2 and is invariant under rotations. The fourth vertex could possibly belong to a Higgs boson with a mass close to that of the $Z^0$, but there can be no more if icosahedral symmetry is satisfied.

This leaves 3 tetrahedra labeled by the 6 synthemes

$$(14, 25, 03), (14, 35, 02); (24, 15, 03), (24, 35, 01); (34, 15, 02), (34, 25, 01) \quad (4)$$

Which carry no charge and are possibly gluon jets $E_{14}, E_{24}, E_{34}$ in 4 space.

The 3 tetrahedra, associated with gluons r,b,g are just the same tritangent rotated in 3 space about the fourth vertex O which could represent a photon. Because $SU(3)$ symmetry dictates only 8 gluon states the ratio of quark to gluon jets is 18/8 as in [10].
4 Dark Matter and Inflation

Figures 2,3 actually lie in the origin of the Clebsch cubic of Fig.4 which is a quantum foam with jets in a 6 dimensional Planck space with jets (cf.[9]). However when 3 quarks are united to form a proton or neutron in 3 space there is an expansion of the order of $10^{20}$ which could account for the massive inflation observed in Planck time just after the Big Bang. This would be accompanied by freezing because of the energy required to make up the mass deficiency. Also Fig.3 clearly shows that anti-quarks lie on a smaller tetrahedron and therefore occupy less space which could account for the predominance of matter over anti-matter. Finally if there is enough heat energy available there could be quarks uniting to create the Dark Matter halos observed around galaxies.

References


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Figure 1: The 5 true and 10 skew cross synthemes
Figure 2: The Coxeter Polytope
Figure 3: Section of Fig. 1
Figure 4: The Clebsch Diagonal Cubic