Fixed Point Theorems in Menger Probabilistic
Quasi-Metric Spaces Using Weak Compatibility

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Abstract

In this paper, we consider Menger probabilistic quasi-metric space and prove a common fixed point theorem for three self maps on this space using the notion of weak compatibility.

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1. Introduction and Preliminaries

K. Menger introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric space has developed in many directions [13]. The idea of K. Menger was to use distribution functions instead of non-negative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. Such a probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological threshold. It is also of fundamental importance in probabilistic functional analysis, non-linear analysis and applications [1, 2, 6, 12].
In this section, some definitions and results on theory of Menger probabilistic quasi metric spaces (briefly, Menger PQM-space) are given to fill in some background for the readers. For further information we refer to [3, 9, 11].

**Definition 1.1 [13].** A mapping \( T : [0, 1] \times [0, 1] \to [0, 1] \) is t-norm if \( T \) is satisfying the following conditions:

(i) \( T \) is commutative and associative;
(ii) \( T(a, 1) = a \) for all \( a \in [0, 1] \);
(iii) \( T(a, b) \leq T(c, d) \) whenever \( a \leq c \) and \( b \leq d \) and \( a, b, c, d \in [0, 1] \).

The following are the basic t-norms:

\[
\begin{align*}
\&T_E(x, y) = \min \{x, y\} \\
\&T_p(x, y) = x \cdot y \\
\&T_L(x, y) = \max \{x + y - 1, 0\},
\end{align*}
\]

Each t-norm \( T \) can be extended [7] (by associativity) in a unique way taking for \( (x_1, \ldots, x_n) \in [0, 1]^n \) the values \( T^1(x_1, x_2) = T(x_1, x_2) \) and \( T^n(x_1, \ldots, x_{n+1}) = T(T^{n-1}(x_1, \ldots, x_n), x_{n+1}) \) for \( n \geq 2 \) and \( x_i \in [0, 1] \), for all \( i \in \{1, 2, \ldots, n + 1\} \).

**Definition 1.2.** A t-norm \( T \) is of Hadžić-type (H-type in short) and \( \mathcal{H} \) if the family \( \{T^n\}_{n \in \mathbb{N}} \) of its iterates defined, for each \( x \) in \([0, 1]\), by \( T^0(x) = 1, T^{n+1}(x) = T(T^n(x), x) \), for all \( n \geq 0 \), is equicontinuous at \( x = 1 \), that is

\[
\forall \varepsilon \in (0, 1) \exists \delta \in (0, 1): x > 1 - \delta \Rightarrow T^n(x) > 1 - \varepsilon \text{ for all } n \geq 1.
\]

There is a nice characterization of continuous t-norm \( T \) of the class \( \mathcal{H}[10] \).

The t-norm \( T_M \) is an trivial example of a t-norm of \( H \)-type, but there are t-norms \( T \) of Hadzic-type with \( T \neq T_M \) (see e.g., [4]).

**Definition 1.3 [4].** If \( T \) is a t-norm and \( (x_1, x_2, \ldots, x_n) \in [0, 1]^n \), then \( T^n_{i=1} x_i \) is defined recurrently by \( 1 \), if \( n = 0 \) and \( T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n) \) for all \( n \geq 1 \). If \( (x_i)_{i \in \mathbb{N}} \) is a sequence of numbers from \([0, 1]\), then \( T^\infty_{i=1} x_i \) is defined as \( \lim_{n \to \infty} T^n_{i=1} x_i \) (this limit always exists) and \( T^\infty_{i=1} x_{i+n} \).

In fixed point theory in probabilistic metric spaces there are of particular interest the t-norms \( T \) and sequence \( (x_n) \subset [0, 1] \) such that \( \lim_{n \to \infty} x_n = 1 \) and \( \lim_{n \to \infty} T^\infty_{i=1} x_{n+i} = 1 \).

**Proposition 1.4 [4, p.39].** If \( T \geq T_L \) then

\[
\lim_{n \to \infty} T^\infty_{i=1} x_{n+i} = 1 \iff \sum_{n=1}^{\infty} (1 - x_n) < \infty.
\]
**Proposition 1.5.** Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of numbers from \([0,1]\) such that \(\lim_{n \to \infty} x_n = 1\) and t-norms \(T\) is of \(H\)-type. Then
\[
\lim_{n \to \infty} T^\infty_{i=n} x_i = \lim_{n \to \infty} T^\infty_{i=1} x_{n+i} = 1.
\]

**Definition 1.6 [9, 11].** A Menger PQM-space is a triple \((X, F, T)\) where \(X\) is a nonempty set, \(T\) is a continuous t-norm and \(F\) is a mapping from \(X \times X\) into \(D_+\) such that, if \(F_{p,q}\) denotes the value of \(F\) at the \((p, q)\), then the following conditions hold:

(PQM1) \(F_{p,q}(t) = F_{q,p}(t) = 1\) for all \(t > 0\) iff \(p = q\).

(PQM2) \(F_{p,q}(t + s) \geq T(F_{p,r}(t), F_{r,q}(s))\) for all \(p, q, r \in X\) and \(t, s > 0\).

On the other hand, since every Menger probabilistic metric space is a Menger PQM-space.

**Definition 1.7 [9, 11].** Let \((X, F, T)\) be a Menger PQM-space.

(i) A sequence \(\{x_n\}\) is said to be \(F\)-convergent to \(x \in X\) if for every \(\epsilon > 0\) and \(\lambda > 0\) there exists \(k \in \mathbb{N}\) such that \(F_{x_n,x}(\epsilon) > 1 - \lambda\) whenever \(n \geq k\).

(ii) A sequence \(\{x_n\}\) in \(X\) is called left Cauchy if for every \(\epsilon > 0\) and \(\lambda > 0\) there is \(k \in \mathbb{N}\) such that \(F_{x_r,x_s}(\epsilon) > 1 - \lambda\) for all \(s \geq r \geq k\).

(iii) A Menger PQM-space \((X, F, T)\) is called left complete if every left Cauchy sequence is \(F\)-convergent to a point in \(X\).

A Menger PQM-space \((X, F, T)\) with the property that every \(F\)-convergent sequence in \(X\) has a unique limit is called a \(F-US\) space.

In 1998, Jungck and Rhoades [5] introduced the following concept of weak compatibility.

**Definition 1.8.** Let \(A\) and \(S\) be mappings from a Menger PQM-space \((X, F, T)\) into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is \(Ax = Sx\) implies that \(ASx = SAx\).

**Lemma 1.9.** Let \((X, F, T)\) be a Menger PQM-space. If there exists \(k \in (0,1)\) such that
\[
F_{p,q}(kt) \geq F_{p,q}(t),
\]
for all \(p, q \in X\) and \(t > 0\) then \(p = q\).
2. Main Results

Theorem 2.1. Let \((X, F, T)\) be a left complete Menger PQM-space and let \(A, B, L : X \to X\) be maps that satisfy the following conditions:

(i) \(T\) is of Hadžić-type;
(ii) Every convergent sequence in \(X\) has a unique limit;
(iii) \(AB(X) \subseteq L(X)\);
(iv) \(L(X)\) is a closed subset of \(X\);
(v) The pair \((L, AB)\) is weakly compatible;
(vi) There is \(\lambda \in (0, 1)\) such that
\[
F_{ABx_{ABy}}(kt) \geq F_{Lx, Ly}(t) \quad \text{for all } x, y \in X \text{ and } t > 0.
\]

Then \(A, B\) and \(L\) have a unique common fixed point.

Proof. Let \(x_0 \in X\). By (iii) we can find \(x_1\) such that \(L(x_1) = AB(x_0)\). By induction, we can find a sequence \(\{x_n\}\) such that \(L(x_n) = AB(x_{n-1})\).

\[
F_{Lx_n, Lx_{n-1}}(k^n t) = F_{ABx_{n-1}, ABx_{n-2}}(k^n t) \\
\geq F_{Lx_{n-1}, Lx_{n-2}}(k^{n-1} t) = F_{ABx_{n-2}, ABx_{n-3}}(k^{n-1} t) \\
\geq F_{Lx_{n-2}, Lx_{n-3}}(k^{n-2} t) \\
\geq \ldots \ldots \geq F_{Lx_0, Lx_1}(t) \quad \text{for all } n \geq 1.
\]

We show that \(\{y_n\}, y_n = L(x_n)\) is a left Cauchy sequence.

Let \(\epsilon > 0\) be given and \(\lambda \in (0, 1)\) be such that \(T^{m-1}(1 - \lambda, \ldots, 1 - \lambda) > 1 - \epsilon\). Also let \(t > 0\) be such that \(F_{y_n, y_{n+1}}(t) > 1 - \lambda\), \(\delta\) be a positive number and \(n_1 \in N\) be such that \(\sum_{n_1}^\infty k^i t \leq \delta\). Then, for every \(n \geq n_1\) and \(m \in N\) we have:

\[
F_{y_n, y_{n+m}}(\delta) \geq F_{y_n, y_{n+m}}(\sum_{n_1}^\infty k^i t) \\
\geq T^{m-1}(F_{y_n, y_{n+1}}(k^n t), \ldots, F_{y_{n+m-1}, y_{n+m}}(k^{n+m-1} t)) \\
\geq T^{m-1}(1 - \lambda, \ldots, 1 - \lambda) > 1 - \epsilon.
\]

Since the space \((X, F, T)\) is left complete. Then there exists \(z \in X\) such that \(\lim_{n \to \infty} L(x_n) = z\), hence \(\lim_{n \to \infty} AB(x_{n-1}) = \lim_{n \to \infty} L(x_n) = z\).
It follows that, there exists \( v \in X \) such that \( L(v) = z \). We prove that \( AB(v) = z \).

Put \( x = x_{2n} \) and \( y = v \) in (vi), we get

\[
F_{ABx_{2n}ABv}(kt) \geq F_{Lx_{2n}Lv}(t)
\]

as \( n \to \infty \), we have

\[
F_{z,ABv}(kt) \geq F_{z,z}(t)
\]

\( F_{z,ABv}(kt) \geq 1 \) for all \( t > 0 \). Hence \( F_{z,ABv}(t) = 1 \). Thus \( z = ABv \).

Since, the pair \((L, AB)\) is weakly compatible, we have \( L(AB(v)) = AB(L(v)) \), hence it follows that \( Lz = ABz \). Now, we prove that \( ABz = z \).

Put \( x = z \) and \( y = x_{2n+1} \) in (vi), we get

\[
F_{ABx_{2n+1}ABx}(kt) \geq F_{Lx_{2n+1}Lx}(t)
\]

as \( n \to \infty \), we have

\[
F_{ABz,z}(kt) \geq F_{ABz,z}(t)
\]

Thus by Lemma 1.9, we have \( z = ABz \). Therefore, \( z = Lz = ABz \). Now, we prove that \( Bz = z \).

Put \( x = Bz \) and \( y = x_{2n+1} \) in (vi), we get

\[
F_{AB(Bz),ABx_{2n+1}ABx}(kt) \geq F_{L(Bz),Lx_{2n+1}Lx}(t)
\]

as \( n \to \infty \), we have

\[
F_{Bz,z}(kt) \geq F_{Bz,z}(t)
\]

Thus by Lemma 1.9, we have \( z = Bz \). We also have \( z = Az \). Therefore, \( z = Az = Bz = Lz \). That is \( z \) is a common fixed point of \( A, B \) and \( L \).

**Uniqueness:** Let \( w (w \neq z) \) be another common fixed point of \( A, B \) and \( L \). Taking \( x = z \) and \( y = w \) in (vi), we have

\[
F_{ABz,ABw}(kt) \geq F_{Lz,Lw}(t)
\]
Thus, by Lemma 1.9, we have $z = w$ and so the uniqueness of the common fixed point.

On taking $B = I$ (identity map) in Theorem 2.1 then we get the following corollary.

**Corollary 2.2.** Let $A$ and $L$ be self maps on a left complete Menger PQM-space $(X, F, T)$ with the properties:

1. $T$ is of Hadžić-type;
2. Every convergent sequence in $X$ has a unique limit;
3. $A(X) \subseteq L(X)$;
4. $L(X)$ is a closed subset of $X$;
5. The pair $(L, A)$ is weakly compatible;
6. There is $k \in (0, 1)$ such that $F_{AX,AY}(kt) \geq F_{LX,LY}(t)$ for all $x, y \in X$ and $t > 0$.

Then $A$ and $L$ have a unique common fixed point.

It should be noticed (see [8, Theorem 3.3] for the case $Ax = x$) that the condition “$T$ is of Hadžić-type” in Corollary 2.2 may be replaced by

$$\lim_{n \to \infty} T_{\mu}^{\infty} F_{LX,AX} \left(\frac{1}{\mu}\right) = 1$$

for some $x \in X$ and some $\mu \in (0, 1)$. Taking into account Proposition 1.4, one obtains the following:

**Corollary 2.3.** Let $(X, F, T_{\mu})$ be a left complete $F$-US Menger PQM-space and let $A, L: X \to X$ be two maps satisfying the following conditions:

1. $A(X) \subseteq L(X)$;
2. $L(X)$ is a closed subset of $X$;
3. There is $k \in (0, 1)$ such that $F_{AX,AY}(kt) \geq F_{LX,LY}(t)$ for all $x, y \in X$ and $t > 0$;
4. The pair $(L, A)$ is weakly compatible.

Then $A$ and $L$ have a unique common fixed point provided that
\[
\sum_{i=1}^{\infty} \left( 1 - F_{L_a, A_b}(\frac{1}{\mu}) \right) < \infty
\]

for some \( x \in X \) and some \( \mu \in (0,1) \).

References


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